MAT 621 Low-dimensional topology

Questions to think about

Some of the questions below are very easy; one or two are probably too hard (though you are encouraged to consult books). I hope they will generate a useful discussion for the class while I'm away (3/16 and 3/18).

Problem 1. Identify the manifolds obtained by surgeries on the unknot with coefficients 0, $+1$, -1, $1/q$, where q is a non-zero integer. In each case, the surgery coefficient is taken with respect to the canonical longitude. Give direct proofs and explicit answers, like S^3 or $S^1 \times S^2$. ($L(1,1)$ is not good enough.)

Problem 2. Show that the following lens spaces are homeomorphic (as oriented manifolds):
(i) $L(p,q)$ and $L(p,q+np)$
(ii) $L(p,q)$ and $L(p,q')$, where $qq' = 1 \mod p$
Also, show that there is an orientation-reversing homeomorphism between $L(p,q)$ and $L(p,-q)$.
These are in fact the only cases when lens spaces are homeomorphic (the proof that there's nothing else is non-trivial; one needs Reidemeister torsions for instance).

Problem 3. Recall that the lens space $L(p,q)$ may be defined as the result of the $-p/q$ surgery on the unknot (with respect to the standard longitude). Show that $L(p,q)$ is homeomorphic, as an oriented manifold, to the quotient space of $S^3 \subset \mathbb{C}^2$ by the action of the cyclic group $\mathbb{Z}/p\mathbb{Z}$, such that the generator of the group sends (z,w) to $(e^{2\pi i/p}z, e^{2\pi iq/p}w)$.
(This is not very easy, unless you are very good at visualizing things. Split S^3 into solid tori along $S^1 \times S^1 \subset \mathbb{C}^2$, where the two S^1 factors are the circles $|z|^2 = 1/2$ and $|w|^2 = 1/2$. Examine the action of $\mathbb{Z}/p\mathbb{Z}$ on each torus and figure out how the corresponding quotients glue up. A detailed proof can be found in Rolfsen.)
If all else fails, at least do the case of $L(2,1)$; then you will show that $L(2,1)$ is homeomorphic to $\mathbb{R}P^3$.

Problem 4. Compute the fundamental group of $L(p,q)$ from the surgery description and the van Kampen theorem. (The quotient description above is another way to compute $\pi_1 L(p,q)$.)

Problem 5. Try to prove that the mapping class group of the torus T^2 is isomorphic to $SL(2,\mathbb{Z})$, the group of 2×2 matrices with integer entries and det $= 1$. Indeed, given $f : T^2 \to T^2$ an orientation-preserving homeo, check that $f_* : \pi_1(T^2) \to \pi_1(T^2)$ is an element of $SL(2,\mathbb{Z})$, and that this gives a group homomorphism. Also, check that this homomorphism is surjective. Checking injectivity is annoying - for a proof, see Rolfsen's book. (One needs to check that a homeo that acts trivially on $\pi_1(T^2)$ is isotopic to the identity.)

By the way, if we use Lickorish's theorem that says that the mapping class group is generated by two "standard" Dehn twists, the statement above is an immediate corollary. Why? What do those Dehn twists correspond to in $SL(2,\mathbb{Z})$? If A, B stand for the right handed Dehn twists around meridian resp. longitude of the torus, check that $(AB)^6 = id$.

Problem 6. Consider the genus g Heegaard decomposition of the sphere, $S^3 = H_1 \cup H_2$, such that H_1 is a "standard" handlebody we usually draw in $\mathbb{R}^3 \subset S^3$ and H_2 is its complement. Describe its gluing map explicitly as a product of Dehn twists from Lickorish's theorem.

Problem 7. (a) Prove that the lens space $L(p,q)$ is the boundary of the 4-manifold obtained from D^4 by attaching 2-handles along the components of the link shown in the picture, with the (integer) framings a_i given by the coefficients of the continued fraction expansion of $-p/q$,

$$-\frac{p}{q} = a_1 - \cfrac{1}{a_2 - \cfrac{1}{\ddots - \cfrac{1}{a_n}}}.$$

Hint: first do the case where the surgery link has two components. Show that regluing the two tori can be interpreted as regluing of a single torus; compose the gluing maps carefully (multiply matrices!) to identify the coefficient of this single surgery. Then proceed by induction. (Details of this can be found, for example, in Saveliev's book.) A slightly different perspective is given by the "slam-dunk" move (Problem 9), from which the required statement follows immediately.

(b) (this part is easier) Use Kirby calculus, rather than the formula from (a), to identify the lens space from the diagram in Figure 1(b) . In general, what happens if one of the coefficients a_i is $+1$ or -1? How does this fit with partial fractions?

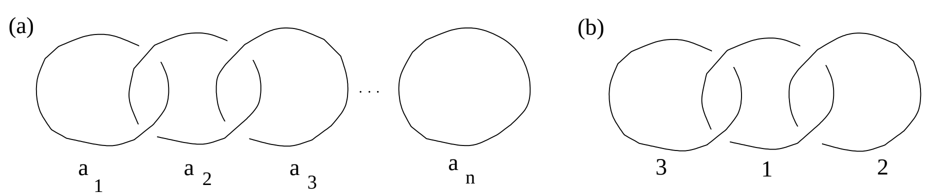

FIGURE 1. (a)To get the lens space described above, perform surgery on these linked unknots, with coefficients $a_1, a_2, \ldots a_n$. (b) What lens space is this? Use Kirby moves.

Problem 8. We discussed Kirby moves in class, and in particular, considered a version of the move shown in Figure 2(i): two strands going through and unknotted circle acquire (or lose) a twist when the circle is moved off the strands.(The -1 framed circle can then be discarded.) We said that the framings would change by -1 if the two strands are from different components of the surgery link. How does the framing change if the two strands are in the same component? Identify the manifold given by surgery diagram in Figure 2(ii).

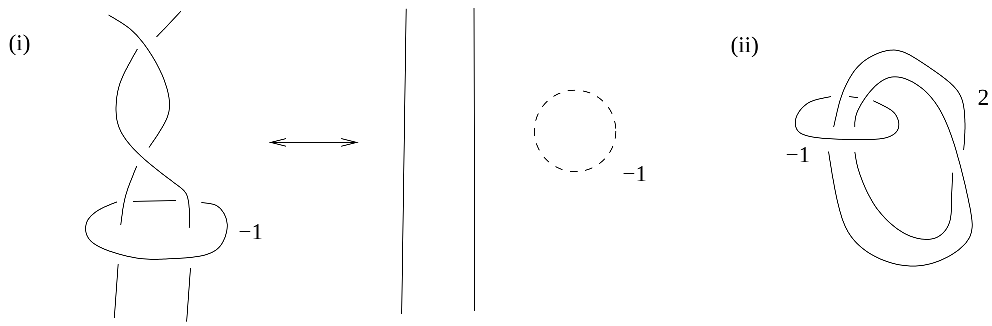

FIGURE 2. (i)Add each of the strands to the circle. As a result, the circle untangles from the strands and can be discarded. If the two strands belong to different components of the link, their framings on the right are 1 less than their framings on the left. (ii) What is this manifold?

Problem 9. Show that the "slam-dunk" move shown in Figure 3 does not change the 3-manifold. (Unlike Kirby moves, it has no 4-dimensional interpretation, but can also be applied to surgeries with rational coefficients.) In the picture, K_1 is the meridian of the another component K_2; the framing of K_2 is $n \in \mathbb{Z}$, the framing of K_1 is $r \in \mathbb{Q}$. The statement is that K_1 can be removed, and surgery coefficient on K_2 then changes from n to $n - \frac{1}{r}$. To see this, consider the manifold obtained by n-surgery on K_2, and let $S^1 \times D^2$ be the solid torus glued in during this surgery. Show that in this surgered manifold, K_1 will be isotopic to the core of this solid torus, ie to the curve $S^1 \times \{pt\}$. Thus the subsequent surgery on K_1 amounts to regluing this solid torus again, so we can think of the surgery on $K_1 \cup K_2$ as just one surgery on K_2. It remains to compute the coefficient. For this, consider the case $n = 0$; then in the torus, meridian goes to longitude, longitude goes to meridian with orientation reversed. In other words, this is a 90 degrees rotation, so the curve with slope r would become the curve with slope $-1/r$. (By the way, in the $n = 0$ case it's easier to see why K_1 is isotopic to the core of the solid torus.) For the arbitrary n case, just add n twists to the longitude of K_2.

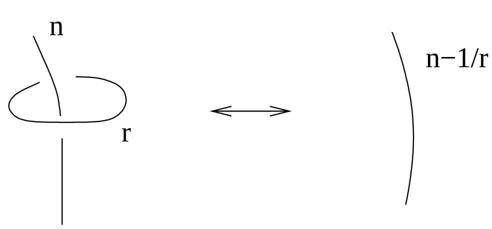

FIGURE 3. The slam-dunk move.