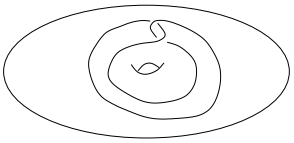
MAT 548, Homework 9, due Friday, Nov 6

1. The Gram-Schmidt othogonalization process can be interpreted to give a deformation retraction of one space of matrices onto another. Give a precise statement (what is a deformation retract of what?) and prove it.

2. Prove that the solid torus $D^2 \times S^1$ does not retract to the curve $C \subset D^2 \times S^1$ shown in the figure below. (Think of $D^2 \times S^1$ as bounded in $\mathbb{R}^3$ by the 2-dimensional torus shown in the figure: the boundary surface of $D^2 \times S^1$ is the 2-torus. The curve is inside.)



3. Let $p : \tilde{X} \to X$ be a covering map, where X and $\tilde{X}$ are path-connected, $x_0 \in X$. Fix $\tilde{x}_0 \in \tilde{X}$, $p(\tilde{x}_0) = x_0$.

We define a (right) action of the fundamental group $\pi_1(X, x_0)$ on the fiber $p^{-1}(x_0)$, as follows. Given $[\gamma] \in \pi_1(X, x_0)$, and an element $\tilde{x} \in p^{-1}(x_0)$, consider the lift $\tilde{\gamma}$ starting at $\tilde{x}$, and set

$$\tilde{x} \cdot [\gamma] = \tilde{\gamma}(1).$$

(a) Show that this map is well-defined (on homotopy classes) and gives a group action.

(b) Show that this action is *transitive*, and that the *stabilizer* of $\tilde{x}_0$ under this action is the subgroup $p_*(\pi_1(\tilde{X}, \tilde{x}_0)) \subseteq \pi_1(X, x_0)$.

Definitions: A group G acts on a set S on the right if every $g \in G$ gives a map $g : S \to S$, written as $x \mapsto x \cdot g$, with the following properties:

- $x \cdot (gh) = (x \cdot g) \cdot h$ for $x \in S$, $g, h \in G$;
- the identity $e \in G$ acts as id_S, that is, $x \cdot e = x$.

An action is transitive if for every $x, y \in S$, there is an element $g \in G$ such that $x \cdot g = y$. The stabilizer of a point $x \in S$ is the subgroup in G that consists of elements fixing x.

Writing the group elements on the right and working with the right action is convenient, because otherwise, path concatenation and composition of homeomorphisms would be written in a different order.

4. Find a 3-fold path-connected covering $p : \tilde{X} \to X$ of the "figure 8" space $X = S^1 \vee S^1$ and a loop $\gamma : I \to X$, $\gamma(0) = \gamma(1) = x_0$, such that when you lift the path γ to $\tilde{X}$, you can get either a closed loop or an open path, depending on its initial point in $\tilde{X}$.

In other words: consider $\tilde{x}_1$, $\tilde{x}_2 \in p^{-1}(x_0)$, and let $\tilde{\gamma}_1$, $\tilde{\gamma}_2$ be two lifts of γ, such that $\tilde{\gamma}_1(0) = \tilde{x}_1$ and $\tilde{\gamma}_2(0) = \tilde{x}_2$. Find an example where $\tilde{\gamma}_1(1) = \tilde{x}_1$, so $\tilde{\gamma}_1$ is a closed loop, but $\tilde{\gamma}_2(1) \neq \tilde{x}_2$ (the path $\tilde{\gamma}_2$ is not a loop).

5. (a) Let X be a path-connected and locally path connected topological space such that $\pi_1(X)$ is finite. Show that every continuous map $f : X \to T^n$ is null-homotopic. (Here, T^n stands for the n-dimensional torus.)

(b) Suppose M is a connected smooth manifold of dimension $m < n$, and $\pi_1(M) = \mathbb{Z}/3$. Show that every map $f : M \to \mathbb{R}\mathrm{P}^n$ is null-homotopic.

6. In class, we discussed the relation of orientation of a smooth manifold to the fundamental group and coverings. This question pins down the details; first we recall the definitions.

Let M be a connected smooth manifold of dimension $m > 0$. For each $x \in M$ there are two choices of orientation $\mathcal{O}_x$ of T_xM. (An orientation of $\mathbb{R}^m$ is an equivalence class of an ordered basis, where the basis $(v_1, \ldots, v_m)$ and the basis $(v'_1, \ldots, v'_m)$ are equivalent if the matrix sending one of them to the other has positive determinant.)

In a chart $U \subset M$ with coordinates $(x_1, \dots x_m)$, we have coordinates $(x_1, \dots, x_m, \frac{\partial}{\partial x_1}, \dots \frac{\partial}{\partial x_m})$ on TU, so that the bundle $TU \to U$ can be identified with $U \times \mathbb{R}^m \to U$. Fixing an orientation of $\mathbb{R}^m$, we then have a choice of orientation $\mathcal{O}_U$ on the chart U. (Here, U can be an interior chart or a boundary chart.)

An orientation on M is a choice of orientation of T_xM for every $x \in M$, such that there is a covering of M by charts with orientations compatible with this choice. The manifold M is called orientable if an orientation on M exists.

Since at every point of M we have two choices of orientations (with no preferred choice), a covering space naturally arises if we consider both choices.
Define

$$\tilde{M} = \{(x, \mathcal{O}_x) \mid x \in M, \mathcal{O}_x \text{ is an orientation of } T_xM\},$$

and let $p : \tilde{M} \to M$ be the projection, $p(x, \mathcal{O}_x) = x$.

To make $\tilde{M}$ a topological space, we need to specify a basis of topology. If U is a chart on M and $\mathcal{O}_U$ is a choice of orientation on U as above, let

$$\tilde{U}_{\mathcal{O}_U} = \{(x, \mathcal{O}_x) \in \tilde{M} \mid x \in U, \quad \mathcal{O}_x \text{ orientation of } T_xM \text{ given by } \mathcal{O}_U\}.$$

(a) Check that the sets $\tilde{U}_{\mathcal{O}_U}$ give a basis of topology on $\tilde{M}$.

(b) Show that $p : \tilde{M} \to M$ is a 2-fold covering. Explain briefly why $\tilde{M}$ has a smooth structure, so that p is a local diffeomorphism between the smooth manifolds $\tilde{M}$ and M.

(c) Show that $\tilde{M}$ is orientable. $\tilde{M}$ is called the *orientation covering* of M.

(d) If M is the Möbius band, what is its orientation covering? Specify the space and the covering map. What is the orientation covering of $\mathbb{R}\mathrm{P}^n$, $n > 1$? (Be careful: the answer depends on the parity of n.)

(e) Prove that M is orientable if and only if $\tilde{M}$ is a trivial 2-fold covering, that is, $\tilde{M}$ is given by two disjoint copies of M, and p restricts to a diffeomorphism on each copy. [M is assumed to be connected.] Conversely, it follows that M is non-orientable if and only if $\tilde{M}$ is connected.

(f) Show that every simply connected manifold is orientable.

7. Informally, a graph is a collection of vertices and edges, so that each edge connects two vertices. (In this question, we allow edges connecting a vertex to itself.) This is a combinatorial notion: to describe a graph, you specify the set of vertices and the pairs that are connected by an edge. Consider only finite graphs for simplicity, although everything works without the finiteness assumption.

A graph becomes a topological space if we think of it as a quotient space of the disjoint union of the closed intervals corresponding to edges, glued so that some of the edges connect at vertices. More formally, start with a finite set of points (vertices) $\{v_1, v_2, \dots, v_k\}$ with discrete topology, and a finite disjoint union of closed intervals (edges) e_j, $1 \le j \le m$. The boundary ∂e_j of the edge e_j is given by the two endpoints of the interval. The endpoints of edges must be glued to vertices of the graph, which means that we are given an attaching map

$$\Phi : \bigsqcup_j \partial e_j \to \{v_1, \dots, v_k\}.$$

Then the graph G corresponding to the above data is the quotient space

$$G = \{v_1, v_2, \dots v_k\} \sqcup \bigsqcup_j e_j \,/(\Phi(x) \sim x, \text{ for all endpoints } x \in \partial e_j, 1 \le j \le m).$$

(a) Prove that every connected graph is path-connected.

(b) Prove that every connected (finite) graph is homotopy equivalent to a wedge of circles. What is the number of circles in the wedge, in terms of the edges and vertices of the graph? What graphs are contractible?