MAT 548, Homework 5, due Friday, Oct 2

From Lee's textbook: **5-17** (more basic technical work with submanifolds) and **6-2** (Whitney's immersion theorem).

More questions:

1. Suppose $F : M \to N$ is a smooth map, and M is compact. Show that regular values of F are open and dense in N.

2. In standard Euclidean space $\mathbb{R}^n$, the translation by a vector $v \in \mathbb{R}^n$ is the map $x \mapsto x + v$.

Let X, Y be two embedded submanifolds without boundary in $\mathbb{R}^n$. Show that there exists a translation τ_v by a vector v of length less than a given $\epsilon > 0$, such that $\tau_v(Y)$ is transverse to X.

The proof can be obtained as a direct application of Sard's theorem (applied to an appropriate map on $X \times Y$), or via parametric transversality. Please do both proofs.

Would you be always able to achieve transversality if you only consider translations by vectors parallel to a given vector v_0? Or by vectors of fixed non-zero length (for example, unit vectors)? Prove or give counterexamples.

Explain briefly how the following corollaries follow from the statement of the problem:

(i) If C_1, C_2 are two simple closed curves in $\mathbb{R}^2$, then after an arbitrarily small translation of one of the curves, the two curves will intersect at most at finitely many points. (A simple closed curve is a compact connected embedded 1-dimensional submanifold without boundary.)

(ii) If C_1, C_2 are two simple closed curves in $\mathbb{R}^3$, then by an arbitrarily small translation of one of the curves, you can make the two curves disjoint.

(iii) If S_1, S_2 are two embedded closed surfaces in $\mathbb{R}^3$, then after an arbitrarily small translation of one of the surfaces, the two surfaces will intersect along a finite collection of simple closed curves, or the intersection will be empty. (The curves may be knotted and linked. A closed surface = compact 2-dimensional manifold without boundary.)

3. (a) Let Γ_1, Γ_2 be two smoothly embedded curves in the unit 3-sphere in $\mathbb{R}^4$. Show that the space

$$Z = \{(A, x, y) \in SO(4) \times \Gamma_1 \times \Gamma_2 : Ax = y\}$$

is a smooth manifold, by representing it as the preimage of the diagonal $\Delta \subset S^3 \times S^3$ under the appropriate map and checking that your map is transverse to Δ.

(b) (August 2026 Comps problem) Let Γ_1, Γ_2 be two smoothly embedded curves in the unit 3-sphere in $\mathbb{R}^4$. Show that for almost all $A \in SO(4)$, we have $A(\Gamma_1) \cap \Gamma_2 = \emptyset$.

There are two ways to solve part (b), either as a direct application of Sard's theorem to a map whose domain is the manifold defined in part (a), or via parametric transversality. Please do both proofs.

More on orientations:

4. Let M be a smooth n-manifold. Show that the smooth $2n$-manifold TM is always orientable.

5. (a) Prove Proposition 5.41 in Lee (this is Exercise 5.42 in the text). Read the discussion in the book surrounding the proposition.

(b) Let M be an orientable smooth manifold with boundary. Prove that its boundary ∂M is orientable.

Note: If M is non-orientable, then ∂M might or might not be orientable. The boundary of the Möbius band is the circle (orientable). On the other hand, the Klein bottle (non-orientable) is a boundary of a (non-orientable) 3-manifold. Try to find this 3-manifold (you don't have to submit this part).