MAT 548, Homework 4, due Friday, Sept 25

From Lee's textbook: please do **questions 4-6, 5-2, 5-6, 5-7, 6-1** at the end of the Chapters 4, 5, 6.

Comments on the questions from Lee:

(i) We focused on the case of manifolds and submanifolds without boundary in class. In particular, Theorem 5.8 assumes that M has empty boundary. Since 5-2 is about a manifold with boundary, you will need to revisit the beginning of Chapter 5 (the definition and Proposition 5.2) which allows for manifolds with boundary. Note that we still assume that submanifolds have empty boundary; we do not (yet) need the material at the very end of Chapter 5. In 6-1, you can also assume that the manifold has empty boundary, although the question works in the same way for manifolds with boundary.

(ii) For 5-7, it may be a bit tricky to prove that some of the level sets are not (embedded) submanifolds. Some helpful tricks might be to reparameterize (set $y = tx$ or $x = ty$) or to use some algebra:

$$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca).$$

More questions:

1. Suppose $F : M \to N$ is a smooth map, where M and N are smooth manifolds of the same dimension, and M is compact.
(a) Explain why $F^{-1}(y)$ is a finite set $\{x_1, \dots, x_k\}$ if y is a regular value of F.
(b) Prove that there exists a neighborhood $V \ni y$ such that $F^{-1}(V)$ is the disjoint union of open sets $U_1, \dots, U_k$, such that $x_i \in U_i$, and $F|_{U_i} : U_i \to V$ is a diffeomorphism for each $i = 1, \dots, k$.

Note: it follows that if N is connected and the map F as above is a submersion, then it is a covering map.

2. In this question, we'll prove the fundamental theorem of algebra: every non-constant complex polynomial $P(z)$ has a root.

(a) We extend $P : \mathbb{C} \to \mathbb{C}$ to a map $f : S^2 \to S^2$, as follows. Recall that the stereographic projection $\sigma : S^2 \setminus \{N\} \to \mathbb{R}^2 = \mathbb{C}$ (see Homework 1, Lee Exercise 1-7) identifies the complement of the North Pole N with the plane, and set $f(x) = (\sigma^{-1} \circ P \circ \sigma)(x)$ for $x \in S^2 \setminus \{N\}$, and $f(N) = N$. Show that the resulting map $f : S^2 \to S^2$ is smooth.
(You can also think of f as extending P to $\mathbb{CP}^1$ via $P(\infty) = \infty$.)

(b) Explain why f has only finitely many critical values.

(c) Using Question 1 and 2(b), prove that for every regular value y of f, the finite set $f^{-1}(y)$ has the same number of elements.

(d) Show that $f^{-1}(y)$ must be non-empty for some regular values y, and therefore it is non-empty for all regular values. Conclude that $f^{-1}(0)$ must be non-empty. (Note that 0 might be a regular or a critical value of f.) Therefore, the polynomial P must have a zero.

3. Let $M(n) = M(n, \mathbb{R})$ be the set of all $n \times n$ matrices with real entries. We identify it with $\mathbb{R}^{n^2}$ as usual.

The orthogonal group $O(n)$ is defined as

$$O(n) = \{A \in M(n) : AA^t = I\},$$

where A^t stands for the transpose of A. It consists of orthogonal matrices, which represent distance-preserving linear transformations of $\mathbb{R}^n$. The special orthogonal group $SO(n)$ consists of $A \in O(n)$ of determinant $\det A = 1$.

(a) Prove that $O(n)$ and $SO(n)$ are compact, $SO(n)$ is path-connected, and $O(n)$ consists of two path-connected components.

(b) Prove that $O(n)$ is a smooth manifold, by representing it as a preimage of a regular value of the function

$$f : M(n) \to S(n), \qquad f(A) = AA^t,$$

where $S(n)$ is the space of symmetric matrices. (Note that $S(n)$ can be identified with the Euclidean space $\mathbb{R}^{n(n+1)/2}$.) To show that I is a regular value of f, compute $D_A f(X)$ directly as

$$D_A f(X) = \lim_{s \to 0} \frac{f(A + sX) - f(A)}{s},$$

and prove surjectivity by manipulating matrices. What is the dimension of $O(n)$ and $SO(n)$?

(c) Show that $SO(2)$ is diffeomorphic to S^1.

Bonus: Show that $O(n)$ and $SO(n)$ are orientable.