MAT 548, Homework 2, due Friday, Sept 11

From Lee's textbook: please do **questions 3-1, 3-5, 3-7, 3-8** at the end of the Chapter 3. For 3-7, you will need to read the small subsection about germs of smooth functions. In 3-8, assume that the manifold has empty boundary (for manifolds with boundary, one needs to be careful setting the conventions for the definition with curves).

1. (a) Find the tangent space at $(1,1,3)$ to the surface $z = x^2 + 2y^2$ in $\mathbb{R}^3$. Use the formula that should be familiar from the multivariable calculus: if $z = f(x,y)$ is the graph of a smooth function, then the tangent plane at (x_0, y_0, z_0) with $z_0 = f(x_0, y_0)$ is given by

$$z - z_0 = \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0).$$

(b) Starting with the abstract definition of the tangent space as the space of derivations at p, show that the formula above is consistent with that definition. Cite specific theorems and lemmas from Lee when you use them. (The graph is endowed with its standard smooth structure as in Lee Example 1.30.)

(c) Let $\Phi(x,y,z) = c$ be a level set of a smooth function $\Phi : \mathbb{R}^3 \to \mathbb{R}$, (x_0, y_0, z_0) a regular point on the surface. Recall from multivariable calculus that the gradient of the function gives the normal vector to the tangent plane. Use this fact to solve the quiz problem: find the tangent space at $(2,1,1)$ to the surface $x^2y + z^3 = 5$ in $\mathbb{R}^3$.

(d) Show that the normal vector recipe from part (c) gives the tangent space defined by the abstract definition. It is convenient to use the "velocity vectors" and equivalence classes of curves definition here (as in Lee Exercise 3-8).

2. (a) Let $F : \mathbb{R}^3 \setminus \{0\} \to \mathbb{RP}^2$ be the projection sending $p = (x,y,z)$ to $[p] = [x : y : z]$. Consider the identity chart on $\mathbb{R}^3 \setminus \{0\}$ and the chart (U, ψ) on $\mathbb{RP}^2$, where $U = \{z \neq 0\}$, $\psi([x : y : z]) = (x/z, y/z) = (u, v)$.

For these charts, we have the coordinate bases

$$\frac{\partial}{\partial x}\Big|_p, \frac{\partial}{\partial y}\Big|_p, \frac{\partial}{\partial z}\Big|_p \text{ in } T_p(\mathbb{R}^3 \setminus \{0\}) \text{ and } \frac{\partial}{\partial u}\Big|_{[p]}, \frac{\partial}{\partial v}\Big|_{[p]} \text{ in } T_{[p]}\mathbb{RP}^2.$$

Find the matrix of d_pF with respect to these bases.

(b) Spherical coordinates! We use the parameterization of $\mathbb{R}^3 \setminus \{0\}$ given by

$$x = r \sin\xi \cos\theta, \quad y = r \sin\xi \sin\theta, \quad z = r\cos\xi$$

with $r > 0$, $\theta \in (-\pi, \pi)$, and $\xi \in (0, \pi)$. Note that the corresponding chart is the inverse of the parameterization. On $\mathbb{RP}^2$, consider the same chart as in (a).

Let $F : \mathbb{R}^3 \setminus \{0\} \to \mathbb{RP}^2$ be the projection map as above. Find $d_pF\left(\frac{\partial}{\partial r}\Big|_p\right)$ without computing anything. Explain briefly.

Find the matrix of d_pF in the bases of $T_p(\mathbb{R}^3 \setminus \{0\})$ and $T_{[p]}\mathbb{RP}^2$ corresponding to the given charts.

3. (Point-set topology review) Prove that every connected topological manifold is path-connected. [Please don't give a sneaky one-line proof that follows from some theorem; give a reasonably self-contained proof that actually explains why this works.]

Reading assignment: We will discuss partitions of unity on Tuesday, but we will only prove their existence in simple cases (eg compact manifolds). The general case is a bit tedious and requires careful work using the second countable property and local compactness to produce a locally finite collection of open sets. Please spend time to work through the details in Lee Chapter 2 (including revelant facts from Appendix A), it is a useful point-set topology exercise.