MAT 548, Homework 10, due Friday, Nov 13

1. Show that the complex projective space $\mathbb{C}\mathrm{P}^n$ is a CW complex. (This requires describing the cellular structure with the attaching maps for the cells.) Compute $\pi_1(\mathbb{C}\mathrm{P}^n)$.

2. Let X be a CW complex. Show that the following properties are equivalent:

(i) X is path connected;

(ii) X is connected;

(iii) The 1-skeleton X^1 is connected.

3. Let X be a CW complex, A its subcomplex. Show that A has an open neighborhood U in X such that U deformation retracts to A. (Note that X and A are not assumed to be finite, and you need to construct the neighborhood and carefully justify continuity of the required maps.)

4. (a) Let X be any topological space, and let $\phi_0, \phi_1 : \partial D^n \to X$ be two homotopic continuous maps, where D^n is the n-disk, $n \geq 1$ Show that

$$X \sqcup_{\phi_0} D^n \sim X \sqcup_{\phi_1} D^n,$$

that is, attaching a cell via two homotopic maps produces homotopy equivalent spaces.

Hint: you can find a space Y such that both of your spaces are subsets of Y, and Y deformation retracts on each of them.

(b) Let X, X' be two homotopy equivalent topological spaces, $f : X \to X'$ a homotopy equivalence. Let $\phi : \partial D^n \to X$ be continuous. Show that

$$X \sqcup_{\phi} D^n \sim X' \sqcup_{f\circ\phi} D^n,$$

more precisely, the map $X \sqcup_{\phi} D^n \to X' \sqcup_{f\circ\phi} D^n$ determined by f and id_{D^n} is a homotopy equivalence.

Note: These two facts are very useful in practice when looking for homotopy equivalences between CW spaces, that is, when X and X' are CW complexes and the maps ϕ, ϕ_i are cell attachments.

5. A *plane triangle* is an unordered triple of non-collinear points in the plane. The space T of all plane triangles is a topological space in a natural way: in the space $(\mathbb{R}^2)^3$ of ordered triples of points, consider the subspace of non-collinear triples, and take the quotient by the action of the symmetric group S_3 permuting the vertices. (You can describe this topology in different ways: informally, you want to say that two triangles are close if their vertices are close.) Let $R \subset T$ be the subspace consisting of right triangles.

(a) Let $t_0 \in R$ be a basepoint. Compute the fundamental groups $\pi_1(T, t_0)$ and $\pi_1(R, t_0)$. Describe the homomorphism $i_* : \pi_1(R, t_0) \to \pi_1(T, t_0)$ induced by the inclusion $i : R \hookrightarrow T$. (It is not hard to see that both T and R are path-connected.)

(b) Does there exist a retraction $r : T \to R$?

6. This question is about spaces obtained as quotients by group actions and about their fundamental group.

Let X be a topological space, G a group acting on X on the left. This means that we are given a homomorphism of the group G into the group $Homeo(X)$ of homeomorphisms of X, that is, every $g \in G$ gives a homeomorphism $g : X \to X$, written as $x \mapsto g \cdot x$, and $gh \cdot x = g \cdot (h \cdot x)$. (See Note 2 below on right vs left actions.)

The *orbit* of $x \in X$ is the set $O_x = \{g \cdot x : g \in G\}$. The space of orbits X/G is the quotient space of X under the equivalence relation $g \cdot x \sim h \cdot x,\ g, h \in G$. We will need some technical properties to ensure that the orbit space is nice:

(*) Each $x \in X$ has a neighborhood U such that all the images of U under $g \in G$ are disjoint: if $g \cdot U \cap h \cdot U \neq \emptyset$, then $g = h$.

(**) For every $x, y \in X$ that are not in the same orbit, there are neighborhoods $U \ni x$ and $V \ni y$, such that $g \cdot U \cap V = \emptyset$ for all $g \in G$.

An action of G on X is called free if non-trivial elements do not have fixed points: if $g \in G$ and $g \cdot x = x$ for some $x \in X$, then $g = e$. A group action satisfying (*) is always free.

(a) Explain why if the action satisfies (**), then X/G is Hausdorff, and if X is compact, then X/G is also compact.

(b) Show that if G is a finite group acting freely on a Hausdorff space X, then the action satisfies (*) and (**). It follows that in this case, X/G is also Hausdorff.

Note 1: If X is a smooth manifold, and the group action satisfies (*) and (**), then X/G is a smooth manifold. Why is it locally Euclidean, and how do you define the smooth structure on X/G? (Do not submit this part.) See Lee, Theorem 21.13.

(c) Consider the group G of plane transformations, generated by the translation $t : (x, y) \mapsto (x, y + 1)$ together with the affine transformation $a : (x, y) \mapsto (x + \frac{1}{2}, -y)$. Show that this action satisfies (*) and (**). What do orbits of this action look like? Identify the quotient space $\mathbb{R}^2/G$ as a familiar (orientable or non-orientable) surface. Justify your answer.

(d) Show that if (*) is satisfied, then $p : X \to X/G$ is a covering map. (If X and the G action are as in Note 1, it will be a smooth covering.)

(e) Prove that if X is simply connected and (*) holds, then $\pi_1(X/G) = G$.
The isomorphism is constructed as follows. Fix $x_0 \in X$, let $\bar{x}_0 = p(x_0) \in X/G$. If γ is a loop in X/G based at $\bar{x}_0$, consider its lift $\tilde{\gamma}$ in X starting at x_0. If $\tilde{\gamma}$ ends at $g \cdot x_0$, set $\phi([\gamma]) = g$.

Explain why $\phi : \pi_1(X/G, \bar{x}_0) \to G$ is well-defined and bijective. Show that ϕ is a group homomorphism.

(f) Explain how part (e) can be used to find the fundamental group of the n-dimensional torus, the n-dimensional real projective space for $n \geq 2$. Find the fundamental group of the surface in part (c) in terms of generators and relations. There are many other important examples, quotient spaces are ubiquitous in mathematics.

Note 2: The difference between right group actions and left group actions is largely notational, but it affects the order of operations, so care is needed to make things consistent (the product in the group and path concatenation in this question). Right group actions can be converted to left group actions (and conversely) by composing with taking inverses in the group.