MAT 548, Homework 1, due Friday, Sept 4

From Lee's textbook: please do **questions 1-7, 1-9, and 2-6** at the end of the chapters.

Read Examples 1.18, 1.34 and Proposition 1.45 about product manifolds, and do **question 2.2** to prove Proposition 2.12.

More questions:

1. Prove that the union of three coordinate axes in $\mathbb{R}^3$ is not a topological manifold. (Please give a rigorous argument using topological properties; intuition should be clear.)

2. Let k be an odd positive integer, $k > 1$. Take the smooth atlas on $\mathbb{R}$ consisting of a single chart $(\mathbb{R}, \phi)$, $\phi(t) = t^{1/k}$.

(a) Show that this chart is not compatible with the chart $(\mathbb{R}, \mathrm{id})$ that defines the standard smooth structure on $\mathbb{R}$, and therefore the given smooth structure is *not the same* as the standard smooth structure on $\mathbb{R}$.

(b) Show, however, that this smooth manifold is *diffeomorphic* to $\mathbb{R}$ with its standard smooth structure.

This tells you that you have to be careful with your equivalence relations: "diffeomorphic" is a different notion from "the same" smooth structure. Usually, one wants to study smooth manifolds up to diffeomorphism.

Note: exotic (i.e. non-diffeomorphic to the standard) smooth structures exist on $\mathbb{R}^n$ only for $n = 4$. Existence for $n = 4$ is a famous result of Donaldson.

3. Let $M(n, \mathbb{R})$ be the set of all $n \times n$ matrices with real entries. By considering the entries as coordinates, we can identify $M(n, \mathbb{R})$ with $\mathbb{R}^{n^2}$ and endow it with the standard topology.

The general linear group $GL(n, \mathbb{R})$ consists of invertible $n \times n$ real matrices. The special linear group $SL(n, \mathbb{R})$ consists of $n \times n$ real matrices of determinant 1.

(a) *Warm-up.* Prove that $GL(n, \mathbb{R})$ is an open subset of $M(n, \mathbb{R})$ that has two path-connected components. Prove that $SL(n, \mathbb{R})$ is a path-connected closed subset.

(b) Using Example 1.32 in Lee, check that $SL(\mathbb{R}, n)$ is a smooth manifold. Indeed, $SL(\mathbb{R}, n) = \det^{-1}(1)$ is a level set for the determinant function $\det : M(n, \mathbb{R}) \to \mathbb{R}$. It remains to check that this level set is *non-critical*, namely $D(det)(A)$ is non-zero whenever $\det A = 1$. For this, first show

$$\left.\frac{d}{dt}\right|_{t=0} \det(I + tX) = \operatorname{tr} X,$$

by expanding $\det(I + tX)$ in powers of t. [This is a very important relation between determinant and trace.] Then, compute $D(\det)(A)$ by multiplying by A^{-1} to reduce to the case $A = I$, and explain the conclusion.

4. The Grassmannian $G(k, n)$ is defined as the space of all k-dimensional linear subspaces of $\mathbb{R}^n$. Real projective spaces are a special case of Grassmannians ($k = 1$). Explain briefly what the natural topology on $G(k, n)$ is (you don't have to check any axioms if you describe it directly; a convenient description involves quotient topology).

Grassmannians are smooth manifolds, which can be seen as follows. The smooth charts consist of subspaces that project isomorphically onto a given k-dimensional subspace of $\mathbb{R}^n$ (coordinate maps can be given via identification with certain sets of matrices).

Work out the details of the definition of the smooth charts, and show that the transition maps are smooth. (You don't have to check other properties.)

This fact is explained in many standard textbooks, including Lee. If using literature, please state so (with a reference), but use your own notation and make a self-contained proof without copying from a book. For concreteness, you can work with $G(2, 4)$ if that's easier for notation and calculations.

Reading assignment: in Milnor's Differential Topology textbook, read Appendix on Classifying one-dimensional manifolds.