MAT 542 Complex Analysis I

Problem Set 6

due Tuesday, March 13

Problem 1. Let f be bounded and holomorphic in $\{z \in \mathbb{C} : |z| > R\}$.
(i) Show that f has a Laurent series representation of the form

$$f(z) = c_0 + \frac{c_1}{z} + \frac{c_2}{z^2} + \cdots$$

containing non-positive powers of z only.
(ii) If $M(r) = \sup_{|z|=r} |f(z)|$, show that $r \mapsto M(r)$ is strictly decreasing in $(R, +\infty)$ unless f is constant.

Problem 2. Let U be a bounded open connected set, $\{f_n\}$ a sequence of functions holomorpic in U and continuous in the closure of U. Assume that $\{f_n\}$ converges uniformly on the boundary of U. Show that $\{f_n\}$ converges uniformly on U.

Problem 3. Let $a_1, a_2, \ldots a_n$ be points on the unit circle. Prove that there exists a point z on the unit circle so that the product of the distances from z to a_j is at least 1.

Problem 4. Suppose that $f(z)$ is holomorphic on the closed unit disk, and $|f(z)| \le M$ for $|z| \le 1$. Let $|f(0)| = a > 0$. Denote by N the number of zeroes of f in the disk $\{|z| < 1/3\}$. Show that

$$N \le \frac{1}{\log 2} \log \frac{M}{a}.$$

Hint: If $z_1, \ldots z_k$ are the zeroes of f in $\{|z| < 1/3\}$, consider the function

$$g(z) = \frac{f(z)}{\left(1 - \frac{z}{z_1}\right)\left(1 - \frac{z}{z_2}\right)\left(1 - \frac{z}{z_k}\right)}.$$

Problem 5. Determine the number of zeroes of the polynomial $2z^5 - 6z^2 + z + 1$ in the annulus $1 \le |z| \le 2$.

Problem 6. Let λ be real, $\lambda > 1$. Show that the equation

$$\lambda - z - e^{-z} = 0$$

has exactly one root in the half-plane $\Re z \ge 0$, and that this root is real.

Problem 7. (i) Let $\{f_n\}$ a sequence of functions which are all holomorphic and injective in the unit disk $\{|z| < 1\}$. Suppose that $\{f_n\}$ converge uniformly in any smaller disk $\{|z| \le r < 1\}$ to a non-constant holomorphic function f. Show that f is also injective in the unit disk.
Remark. We will show very soon that a uniform limit of holomorphic functions is necessarily holomorphic, so the assumption that f is holomorphic is actually extraneous.