MAT 542 Complex Analysis I

Problem Set 5

due Tuesday, March 6

Problem 1. Let f be an entire function and let $a, b \in \mathbb{C}$ such that $|a|, |b| < R$. If γ is a circle of radius R, evaluate

$$\int_\gamma \frac{f(z)}{(z-a)(z-b)}\, dz.$$

Use this result to give another proof of Liouville's Theorem.

Problem 2. When computing the integrals below, please explain everything you are doing. Just plugging the function into "a theorem from Lang" will receive little credit. (You may use Jordan's lemma without proof).

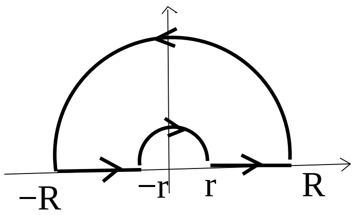
(a) Show that

$$\int_0^\infty \frac{x^a}{x(1+x^3)}\, dx = \frac{\pi}{3\sin(\pi a/3)} \text{ for } 0 < a < 3.$$

(b) Show that

$$\int_0^\infty \frac{\log x}{(x^2+1)^2}\, dx = -\pi/4.$$

Hint: Integrate over contour shown below; use the branch of $\log z$ defined in the domain $\{z \neq 0, -\pi/2 < \arg z < 3\pi/2\}$ by $\log z = \log|z| + i\phi$ for $z = |z|e^{i\phi}$ with $-\pi/2 < \phi < 3\pi/2$.



Problem 3. Let $\gamma = [n + \frac{1}{2} + ni, -n - \frac{1}{2} + ni, -n - \frac{1}{2} - ni, n + \frac{1}{2} - ni, n + \frac{1}{2} + ni]$ be the rectangular path (whose corners are given), and evaluate the integral

$$\int_\gamma \frac{\pi \cot \pi z}{(z+a)^2},$$

where a is NOT an integer.

Show that

$$\lim_{n\to\infty} \int_\gamma \frac{\pi \cot \pi z}{(z+a)^2} = 0.$$

(**Hint:** Use the fact that for $z = x + iy$, $|\cos z|^2 = \cos^2 x + \sinh^2 y$ and $|\sin z|^2 = \sin^2 x + \sinh^2 y$ to show that $|\cot \pi z| \leq 2$ for z on γ if n is large enough.)

Deduce that

$$\frac{\pi^2}{\sin^2 \pi a} = \sum_{n=-\infty}^{\infty} \frac{1}{(a+n)^2}.$$

Find

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}.$$

Problem 4. Suppose that $f(z)$ is holomorphic on and inside the unit circle, $|z| \leq 1$, except for one singular point z_0 on the circle. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be the power series for f on the unit disk. Prove that if f has a pole at z_0, then

$$\lim_{n \to \infty} \frac{a_n}{a_{n+1}} = z_0.$$

(a) Prove this statement for the case when f has a *simple* pole at z_0 (i.e. a pole of ord=1).

(b) Prove this statement in the general case.

Problem 5. Suppose that $f(z)$ is holomorphic on and inside the unit circle, $|z| \leq 1$, except for the point $z = 1$ where f has a pole. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be the power series for f on the unit disk. Show that $\sum_{n=0}^{\infty} a_n z^n$ cannot converge at any point of the unit circle.

The point $z = 1$ has no special significance and is chosen for convenience. The statement is true for any function with exactly one pole on the unit circle (and no other singularities).