MAT 542 Complex Analysis I

Problem Set 4

due Tuesday, February 27

Problem 1. Prove the claim that we needed for the Maximum Modulus Principle:
Let $U \subset \mathbb{C}$ be open, connected, $f : U \to \mathbb{C}$ holomorphic. Suppose that all the values of f lie on the unit circle, i.e. $|f(z)| = 1$ for all z. Then f is constant.

Problem 2. Suppose f and g are holomorphic in a connected open set $U \subset \mathbb{C}$, and $|f(z)|^2 + |g(z)|^2 = 1$ for all $z \in U$. Show that f and g are constant.

Problem 3. A function $f : \mathbb{C} \to \mathbb{C}$ is called *doubly periodic* if there are non-zero complex numbers α, β, with $\alpha/\beta \notin \mathbb{R}$, such that

$$f(z + \alpha) = f(z + \beta) = f(z) \qquad \text{for all } z \in \mathbb{C}.$$

Show that every doubly periodic entire function is constant.

Problem 4. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function, N a positive integer, $C > 0$.
(a) Suppose that $|f(z)| \leq C|z|^N$ for all $z \in \mathbb{C}$. Show that f is a polynomial.
(b) Suppose that $|f(z)| \geq C|z|^N$ for all $|z|$ large enough. Show that f is a polynomial.

Problem 5. Consider the function

$$f(z) = 1 + z^2 + z^4 + z^8 + \dots, \qquad |z| < 1.$$

(Check that the radius of convergence is 1.)
Show that for every $a = e^{k \cdot 2\pi i/2^n}$, $n \geq 0, k = 1, 2, \dots 2^{n-1}$, $f(z)$ tends to ∞ when z approaches a along the radius of the circle, $\lim_{r>0, r\to 1-} f(r \cdot a) = \infty$. Conclude that every point of the unit circle is a (non-isolated) singularity for f.

Problem 6. Suppose that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ has radius of convergence equal to 1, and there are only poles of first order on the unit circle (no other singularities). Show that the sequence $\{a_n\}$ is bounded.

Problem 7. Suppose you want to represent the function

$$f(z) = \frac{1}{1 - z^2} + \frac{1}{3 - z}$$

by a Laurent series $\sum_{j=-\infty}^{\infty} a_j \, z^j$. How many such representations are there? In which region is each of them valid? Find the coefficients a_j explicitly for each of these representations.