MAT 542 Complex Analysis I

Problem Set 1

due Tuesday, February 6

Problem 1. (i) Let $f(z) = 1/z$ (this map is called **inversion** through unit circle). Describe what f does to the inside and outside of the unit circle, and also what it does to points on the unit circle.

(ii) Let $f(z) = 1/\bar{z}$ (this map is called **reflection** through the unit circle). Describe f in the same manner as in (i).

Problem 2. (Gauss-Lucas) Recall that the **convex hull** of a set $\{z_1, \dots, z_k\}$ in $\mathbb{C}$ consists of all $z \in \mathbb{C}$ which can be written as $z = \sum_{j=1}^k \lambda_j z_j$ for some $0 \le \lambda_j \le 1$ such that $\sum_{j=1}^k \lambda_j = 1$. If P is a complex polynomial, show that the roots of the derivative P' belong to the convex hull of the roots of P. (**Hint:** Write $P(z) = \prod_{j=1}^k (z - z_j)^{m_j}$, so that

$$\frac{P'(z)}{P(z)} = \sum_{j=1}^k \frac{m_j}{z - z_j} = \sum_{j=1}^k \frac{m_j(\bar{z} - \overline{z_j})}{|z - z_j|^2}.$$

If $P'(z) = 0$ but $P(z) \neq 0$, then

$$z \sum_{j=1}^k \frac{m_j}{|z - z_j|^2} = \sum_{j=1}^k \frac{m_j z_j}{|z - z_j|^2}.)$$

Problem 3. Assume that the function $f : \mathbb{C} \to \mathbb{C}$ is holomorphic. Let the functions $u, v : \mathbb{R}^2 \to \mathbb{R}$ be the real and imaginary parts of f, so that $f = u + iv$. Show that if $u = v^2$ everywhere, then f is constant.

Problem 4. Given $f : D \to \mathbb{C}$, consider f as a function $D \to \mathbb{R}^2$, and let

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right), \qquad \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right).$$

Show that $\frac{\partial z}{\partial \bar{z}} = 0$, $\frac{\partial \bar{z}}{\partial \bar{z}} = 1$, $\frac{\partial z}{\partial z} = 1$, $\frac{\partial \bar{z}}{\partial z} = 0$.

Show that f is holomorphic in D if and only if $\frac{\partial f}{\partial \bar{z}} = 0$.

Consider $g(z) = z\bar{z} = |z|^2$. Show that g is differentiable (in the complex sense) only at $z = 0$. (It follows that g is not holomorphic at 0, because there's no open neighborhood of 0 where f is differentiable).

Problem 5. Give an example of a power series whose radius of convergence is 1, and such that the corresponding function is continuous on the closed unit disk. (**Hint:** Try $\sum z^n/n^2$.)

Problem 6. Suppose that the power series $\sum a_n z^n$ and $\sum b_n z^n$ both have radius of convergence $R > 0$. Then we have analytic functions

$$f(z) = \sum a_n z^n \quad \text{and} \quad g(z) = \sum b_n z^n$$

in $D_R(0)$. Define the sequence (c_n) by

$$c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0, \quad n = 0, 1, 2 \ldots .$$

Show that the series $\sum c_n z^n$ converges in $D_R(0)$ and therefore defines an analytic function $h(z)$. Prove that $h(z) = f(z)g(z)$ in $D_R(0)$.
Can $\sum c_n z^n$ have a larger radius of convergence?

Problem 7. (i) For any sequence $\{a_j\}$ of non-zero complex numbers, show that

$$\liminf_{j\to\infty} \left|\frac{a_{j+1}}{a_j}\right| \leq \liminf_{j\to\infty} \sqrt[j]{|a_j|} \leq \limsup_{j\to\infty} \sqrt[j]{|a_j|} \leq \limsup_{j\to\infty} \left|\frac{a_{j+1}}{a_j}\right| .$$

(ii) Conclude that if $\{a_j\}$ is a sequence of non-zero complex numbers for which $L := \lim_{j\to\infty} |a_{j+1}/a_j|$ exists, then the radius of convengence of the power series $\sum a_j\, z^j$ is $1/L$.
(iii) Give an example of a sequence $\{a_j\}$ of positive numbers such that

$$\liminf_{j\to\infty} \frac{a_{j+1}}{a_j} = 0,\ \liminf_{j\to\infty} \sqrt[j]{a_j} = 1,\ \limsup_{j\to\infty} \sqrt[j]{a_j} = 2,\ \limsup_{j\to\infty} \frac{a_{j+1}}{a_j} = +\infty.$$

This shows that the **ratio test** in (ii) does not always give the right radius of convergence.

Problem 8. Let $f(z) = \sum a_n z^n$ be an analytic function in $D_R(0)$. We say that $f(z)$ is even if $f(z) = f(-z)$, and $f(z)$ is odd if $f(z) = -f(-z)$ for all z. Show that f is even if and only if $a_n = 0$ for n odd, and that f is odd if and only if $a_n = 0$ for n even.