MAT 542 Complex Analysis I

Problem Set 10

due Tuesday, May 1

Problem 1. Let f and g be entire functions with $f(0) = g(0)$, and let P and Q be polynomials. Assume

$$e^{f(z)} + P(z) = e^{g(z)} + Q(z) \qquad \text{for all } z \in \mathbb{C}.$$

Show that $f = g$ and so $P = Q$. (**Hint:** Write $P - Q = e^g(1 - e^{f-g})$. Use Picard's Theorem to show that the function $1 - e^{f-g}$ is either constant or takes the value 0 infinitely often.)

Problem 2. Give an example of a harmonic function $u : V \to \mathbb{C}$ (where V is open, connected, but not simply-connected) such that u cannot be written as $\operatorname{Re} f$ for any f holomorphic in V.

Problem 3. Let $U \subset \mathbb{C}$ be a domain and $E \subset U$ be a set which has an accumulation point in U. Recall that by the Uniqueness Principle, if f, g are holomorphic in U and $f = g$ on E, then $f = g$ in U.

Show that the analogue of the Uniqueness Principle for harmonic functions fails. In other words, find U, E and harmonic functions $\varphi, \psi : U \to \mathbb{C}$ such that $\varphi = \psi$ on E but $\varphi \neq \psi$ in U.

Show however that if $E \neq \emptyset$ is open, $\varphi, \psi : U \to \mathbb{C}$ are harmonic, and $\varphi = \psi$ on E, then $\varphi = \psi$ in U. (**Hint:** Let $V \supset E$ be the largest open subset of U in which $\varphi = \psi$. If $V \neq U$, choose $p \in \partial V \cap U$ and a disk $D_r(p) \subset U$, and show that the harmonic function $\varphi - \psi$ must vanish in $D_r(p)$. This contradicts $p \in \partial V$.)

Problem 4. This question establishes the definition and properties of an important meromorphic function $\Gamma(z)$ (the gamma function).

(i) Let $z \in \mathbb{C}$, $\operatorname{Re} z > 0$. Define

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt,$$

where $t^{z-1} = e^{(z-1)\log t}$, and we choose a branch of log which gives the real-valued $\log t$ for $t > 0$. Show that this improper integral converges.

(ii) Show that $\Gamma(z)$ is holomorphic in $\{\operatorname{Re} z > 0\}$. (**Hint:** Consider functions $F_n(z) = \int_{1/n}^\infty t^{z-1} e^{-t}\, dt$, $\operatorname{Re} z > 0$, $n = 1, 2, \ldots$ Show that F_n is holomorphic in $\{\operatorname{Re} z > 0\}$ for all n.)

(iii) Integrate by parts to prove that

$$\Gamma(z+1) = z\Gamma(z). \qquad (*)$$

In particular, show that $\Gamma(n+1) = n!$ for $n = 0, 1, 2, \ldots$.
Explain how $(*)$ can be used to get an analytic continuation of Γ into $\mathbb{C} - \{0, -1, -2, -3, \ldots\}$.
Show that $\Gamma(z)$ has poles at $0, -1, -2, \ldots$.
(iv) For a different way to get the analytic continuation, show that for $\operatorname{Re} z > 0$

$$\Gamma(z) - \sum_{k=0}^{n} \frac{(-1)^k}{k!} \frac{1}{z+k} = \int_0^1 \left(e^{-t} - \sum_{k=0}^{n} \frac{(-1)^k}{k!} t^k \right) t^{z-1}\, dt + \int_1^{\infty} e^{-t} t^{z-1}\, dt.$$

Arguing as in (i) and (ii), show that this equality gives an analytic continuation of $\Gamma(z) - \sum_{k=0}^{n} \frac{(-1)^k}{k!} \frac{1}{z+k}$ into the half-plane $\{\operatorname{Re} z > -(n+1)\}$, so that $\Gamma(z)$ becomes a meromorphic function in this half-plane. Conclude that $\Gamma(z)$ is meromorphic in $\mathbb{C}$ with poles of order 1 at $0, -1, -2, \ldots$, and find residues at all these poles.