MAT 541, Algebraic Topology

Homework 1

If you are not very familiar with the notions of homotopy type and homotopy equivalence, as well as basic operations on topological spaces, and the basic definition of CW complexes, please read Chapter 0 of Hatcher (p.1-14 only; skip homotopy extension preperties for now). CW complexes will be discussed in class later, in some more detail.

Please do exercises 5, 9, 10, 19, 23 on p.18-20.

Now for homology:

Question 0. Make sure you have a solid working knowledge of the basic homological algebra of chain complexes and exact sequences. For example, you should be able to prove the 5-lemma and to produce proofs of exactness (at every term) of the long exact sequence of the pair. If you are writing up any of these proofs as homework, please include diagrams illustrating the diagram-chasing you are doing.

Question 1. State and prove the long exact sequence of a pair for simplicial homology.

Question 2. Prove the long exact sequence of a triple, both for simplicial and singular homology. (We stated it in class but didn't prove.) Describe all the maps in the sequence explicitly.

Question 3. State and prove excision theorem for simplicial homology, in the formulation that relates homology of the pair $(A \cup B, A)$ to homology of the pair $(B, A \cap B)$. Your proof should be easy, much easier than for singular homology!

Question 4. Prove that if $f, g : (X, A) \to (Y, B)$ are homotopic maps of pairs, they induce the same map $f_* = g_* : H_n(X, A) \to H_n(Y, B)$. (This is Proposition 2.19 of Hatcher and is a quick corollary of the constructions in the proof of Theorem 2.10, but it's useful to take a closer look and make sure you understand how it works. Do not submit anything for this question).

Question 5. (a) Let M be a closed oriented smooth manifold of dimension n, and assume that M is triangulated (ie you have a finite simplicial complex whose total space in M). Explain how orientation of M gives "compatible" orientations of all n simplices of M. Prove that $H_n(M) = Z$. What is the generator?

(b) Let M be an oriented compact smooth manifold of dimension n with non-empty boundary ∂M, triangulated so that ∂M is a simplicial subcomplex of M. Prove that $H_n(M) = 0$ and $H_n(M, \partial M) = \mathbb{Z}$.

Most of the questions above can be found either in Hatcher or in other sources, it's okay to consult them if you get stuck; the goal is to understand the proofs well.

The following questions from Hatcher 2.1 (p.131-133) are useful: 1-4, **5**, 7-8, **9**, 10, 11-13, **15-17**, **22**, **30**. Numbers in bold are particularly recommended if you are submitting a written homework.

Of course you don't have to write, and submit all these questions even if you need a grade. Rather, treat the above as the checklist of skills you should acquire in this course – you should be able to solve most of the questions without much trouble.

If working for a grade, please submit (or just come and discuss with me) some substantial amount of work. Be reasonable about which questions you choose – for example I don't need proofs of the 5-lemma copied from the textbook, and you don't have to submit Question 0 if you are very comfortable with homological algebra. But if you are new to exact sequences, please chase a diagram or two.