MAT 540, Homework 9, due Thursday, Nov 1

0. [Do not submit.] In class, we stated that the fundamental group of a wedge sum of spheres of arbitrary dimension is the free group on generators corresponding to the circles in the sum. We proved the statement for a wedge of circles. The general statement can be proved in a similar way, using the fact that the wedge of higher-dimensional spheres (no circles) is simply connected [can you prove this?] and constructing a simply-connected covering of a wedge that includes circles [what does the covering look like?] We'll prove a more general theorem very soon; for now, you can use this fact in the homework.

Go back to Homework 8 and compute π_1 for all the spaces from Question 1 HW 8.

1. Prove the following very easy but very useful facts. Let X be a topological space.

(a) If X is simply connected, $a, b \in X$, then any two paths in X connecting a and b are homotopic. (Be careful: the homotopy must be basepoint-preserving).

(b) X is contractible if and only if id_X is homotopic to a constant map.

(c) A path-connected space X is contractible if and only if any two maps $f, g : X \to Y$ are homotopic, for every path connected space Y.

By definition, a contractible space is homotopy equivalent to a point. Parts (b) and (c) give equivalent definitions.

2. Compute the fundamental group of the following spaces:

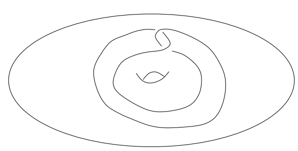
(a) the space $GL_+(2, \mathbb{R})$ of the real 2×2 matrices with positive determinant;

(b) the space $SL(2, \mathbb{R})$ of the real 2×2 matrices with determinant 1;

(c) the space P of quadratic polynomials of the form $p(x) = x^2 + ax + b$, with complex coefficients a, b, such that p has two distinct complex roots (P has induced topology as a subset of $\mathbb{C}^2$, where the coordinates are given by coefficients a, b);

(d) the subspace X in R^3 which is the union of eight spheres centered at the vertices of a cube in $\mathbb{R}^3$, with radius equal half the edge length of the cube.

3. (a) Prove that the solid torus $D^2 \times S^1$ does not retract to the curve $C \subset D^2 \times S^1$ shown in the figure below. (Think of $D^2 \times S^1$ as bounded in $\mathbb{R}^3$ by the 2-dimensional torus shown in the figure: the boundary surface of $D^2 \times S^1$ is the 2-torus. The curve is inside.)



(b) Let $T = S^1 \times S^1$ be a torus, let D be a small disk in T, and let S be the boundary of this disk. Consider the surface with boundary M which is the complement in T of the interior of D (that is, we cut out the open disk from T). Show that M does not retract onto its boundary curve S.

4. If X is path-connected, $p : \tilde{X} \to X$ a covering, and $x_1, x_2 \in X$, show that there is a (non-canonical) bijection between sets $p^{-1}(x_1)$ and $p^{-1}(x_2)$. In particular, if $p^{-1}(x_1)$ is finite and consists of n points, then every fiber consists of n points, and the covering is n-sheeted.

5. This question is about elementary graph theory and topology. Informally, a graph is a collection of vertices and edges, so that each edge connects two distinct vertices. (This is a combinatorial notion: to describe a graph, you specify the set of vertices and the pairs that are connected by an edge.) In this question, we only consider finite graphs for simplicity, although everything works without the finiteness assumption.

A graph becomes a topological space if we think of it as a quotient space of the disjoint union of the closed intervals corresponding to edges, glued so that some of the edges connect at vertices. More formally, start with a finite set of points (vertices) $\{v_1, v_2, \dots v_k\}$ with discrete topology, and a finite disjoint union of closed intervals (edges) e_j, $1 \le j \le m$. The boundary ∂e_j of the edge e_j is given by the two endpoints of the interval. The endpoints of edges must be glued to vertices of the graph, which means that we are given an attaching map

$$\Phi : \bigsqcup_j \partial e_j \to \{v_1, \dots, v_k\}.$$

Then the graph G corresponding to these data is the quotient space

$$G = \{v_1, v_2, \dots v_k\} \sqcup \bigsqcup_j e_j \,/ (\Phi(x) \sim x, \text{ for all endpoints } x \in \partial e_j, 1 \le j \le m).$$

Here are some of the combinatorial notions we need. (The definitions below only use combinatorics rather than topology. If you've never seen them before, you might want to read up on these basics a bit, or you can prove the fact about maximal trees yourself.)
Definition: A *cycle* in G is a sequence of *distinct* edges $e_{i_1}, e_{i_2}, \dots e_{i_k}$ such that e_{i_r} connects to $e_{i_{r+1}}$ at one of its endpoints and to $e_{i_{r-1}}$ at the other, for $1 \le r \le k-1$, and e_{i_k} connects to $e_{i_{k-1}}$ and to e_{i_1}.
Definition: a *tree* is a connected graph with no cycles.
Fact: Every connected finite graph G has a *maximal tree*, which is a subgraph $T \subset G$ such that T is a tree and T contains all the vertices of G (and some of the edges of G). Note that a maximal tree is not unique. [You can use this fact in this question.]

Now we consider a graph as a topological space. Assume that all graphs in (a)-(d) are connected.

(a) Prove that every connected graph is path-connected.

(b) Prove that every (finite) tree is contractible.

(c) Prove that any (finite) graph is homotopy equivalent to a wedge of circles.

(d) Compute the fundamental group of a finite graph in terms of the number of its edges and vertices. As a special case, you will also get a relation between the number of edges and vertices of a tree.