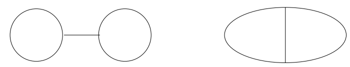
MAT 540, Homework 8, due Thursday, Oct 24

0. *(do not submit but make sure you understand this)*
The two subspaces of $\mathbb{R}^2$ shown below are homotopy equivalent, as can be seen, for example, by representing them as a deformation retract of a third space. You can also show that each of them is homotopy equivalent to the wedge sum of two circles. Explain how to construct explicit homotopy equivalences between these spaces (and why they are homotopy inverses of one another). Arguing by picture is fine but you need to understand what the maps and the homotopies do.



1. The goal of this question is to develop more intuition about homotopy equivalence.
Each of the topological spaces below is homotopy equivalent to a (finite) wedge sum of some spheres, possibly of different dimensions. (Some of the spaces may be homotopy equivalent to a single sphere or a circle, or be contractible.) Recall that for any collection of spaces (X_α, x_α) with chosen basepoints, the *wedge sum* $\vee_\alpha X_\alpha$ is the quotient of the disjoint union $\sqcup_\alpha X_\alpha$ obtained by identifying all $x_\alpha \in X_\alpha$ to a single point.

Find the corresponding wedge sum in each case, and explain briefly what the homotopy equivalences look like and what the required homotopies do. Arguing informally ("by picture") is fine; no formulas are required. Sometimes you will be able to find deformation retracts that are wedges of spheres.

(a) the complement $T^2 \setminus \{p\}$ of a point p in a torus T^2;

(b) the complement of k lines through the origin in $\mathbb{R}^n$;

(c) the union $S^2 \cup D$ of the sphere $S^2 = \{x^2+y^2+z^2 = 1\} \subset \mathbb{R}^3$ and the disk $D = \{x^2+y^2 \leq 1, z = 1\}$.

(d) the union $S^2 \cup I$ of the sphere S^2 in $\mathbb{R}^3$ and a segment I connecting two opposite points of S^2;

(e) the sphere S^n, $n \geq 2$, with k distinct points identified, that is, the quotient S^n/K for $K = \{x_1, x_2, \dots x_k\} \subset S^n$.

Note: Different wedge sums of spheres are not homotopy equivalent (although we cannot prove it yet), so the answer is unique in each case. In particular, keep in mind that higher-dimensional spheres are *not* contractible, even if they are simply connected!

2. The Gram-Schmidt othogonalization process can be interpreted to give a deformation retraction of one space of matrices onto another. Give a precise statement (what is a deformation retract of what?) and prove it.

3. Show that a homotopy equivalence $f : X \to Y$ induces a bijection between the set of path components of X and the set of path components of Y.

4. The *cone* CX over a topological space X is defined as $X \times I/(X \times \{0\})$, where $I = [0, 1]$. The point of CX obtained by collapsing $X \times \{0\}$ is called the vertex of the cone; $X \times \{1\}$ is called the base of the cone. (The base can be identified with X.)

(a) What is CS^n?

(b) Prove that the base of a cone CX is a retract of CX if and only if X is contractible.

5. (a) Let $F : \mathbb{R}^m \to \mathbb{R}^n$ be a smooth map such that $F(0) = 0$. Define

$$H(x,t) = \begin{cases} \frac{1}{t}F(tx), & 0 < t \leq 1 \\ D_0F(x), & t = 0. \end{cases}$$

Show that H is a smooth homotopy between F and the linear map D_0F.

(b) Let $B_r(0) \subset B_R(0)$ be the balls of radius r resp. R centered at 0, $R > r > 0$. Using bump functions or partition of unity, construct a smooth homotopy $H(x,t)$ giving the map $F(x)$ at time $t = 1$ and such such that:

(i) $H(x,t) = F(x)$ for all $0 \le t \le 1$, $x \in \mathbb{R}^m \setminus B_R(0)$;

(ii) $H(x,0) = D_0F(x)$, $x \in B_r(0)$.

This homotopy makes the given map linear in a small neighborhood of 0, while keeping F fixed outside of a larger given neighborhood.

(c) Suppose that $m = n$ and D_0F is non-singular. Suppose that $\det D_0F > 0$. Using (a) and (b), construct a (smooth) homotopy $H(x,t)$ such that

(i) $H(x,t) = F(x)$ for all $0 \le t \le 1$, for all x outside of a given neighborhood of 0;

(ii) $H(x,0) = x$ for x in a small neighborhood of 0.

State and prove a similar result for the case $\det D_0F < 0$.

Note: working in a chart, you can use the result of this exercise to linearize a smooth map between manifolds in a neighborhood of a given point by a homotopy. This is a very useful technical trick (especially when D_0F is non-singular).

6. This question is about spaces obtained as quotients by group actions. We had an intro discussion in class (the summary of definitions is below). We considered the situation where the group G acts on a smooth manifold M by diffeomorphisms, and explained that the orbit space M/G is locally Euclidean and has smooth charts if the action satisfies condition (*) below. (We explained the smooth structures but skipped checking point-set topoogical properties.) Here, you are asked to check some of the topological properties and work out a useful example.

Let X be a topological space, G a group acting on X. This means that we are given a homomorphism of the group G into the group $Homeo(X)$ of homeomorphisms of X: for every $g \in G$, there is a homeomorphism $\phi_g : X \to X$ such that $\phi_{gh} = \phi_h \circ \phi_g$, $g, h \in G$. We will usually write g for the homeomorphism ϕ_g to simplify notation.

The *orbit* of $x \in X$ is the set $O_x = \{g(x) : g \in G\}$. The space of orbits X/G is the quotient space of X under the equivalence relation $g(x) \sim h(x), g, h \in G$, with quotient topology. We will need the following property to ensure that the orbit space is nice.

(*) Each $x \in X$ has a neighborhood U such that all the images $g(U)$ are disjoint, $g \in G$: if $g(U) \cap h(U) \neq \emptyset$, then $g = h$.

An action with property (*) is sometimes called *properly discontinuous*; the terminology differs in the literature. A *free* action of G on X is an action where non-trivial elements do not have fixed points: if $g \in G$ and $g(x) = x$ for some $x \in X$, then $g = e$. A group action satisfying (*) is always free.

(a) If X is Hausdorff and the action satisfies (*), show that each orbit O_x is closed.

(b) Assume that X is compact and Hausdorff. Show that if (*) is satisfied, then X/G is compact and Hausdorff.

(c) Show that if G is a finite group acting freely on a Hausdorff space X, then the action satisfies (*). In this case, X/G is also Hausdorff.

(d) Consider the group G of plane transformations, generated by the translation $t : (x,y) \mapsto (x, y+1)$ of $\mathbb{R}^2$ together with the affine transformation $a : (x,y) \mapsto (x + \frac{1}{2}, -y)$. Show that this action satisfies (*). What do orbits of this action look like? Identify the quotient space $\mathbb{R}^2/G$ as a familiar (orientable or non-orientable) surface. Justify your answer.

Note: although (b) and (c) don't cover examples such as (d), the Hausdorff property for the orbit space X/G will hold if one adds another hypothesis to (*). That would cover the interesting examples that we care about. I'll post a reference once the homework is collected.