MAT 540, Homework 14, due Monday, DEC 9 by the end of the day (note special date)

0. [Do not submit.] Go back to Question 4 Homework 7 and explain how to prove the result using cellular approximation. (You may assume that the smooth manifold is triangulated.)

1. Let X be a CW complex, A is subcomplex (possibly infinite dimensional). We defined a subcomplex as a closed subspace in X which is a union of cells. Check that A is a CW complex, and that the topology from the CW structure is the same as subspace topology on A.

2. Let X be a CW complex, A and B its subcomplexes, $X = A \cup B$. Suppose that A and B are both contractible, and $A \cap B$ is non-empty and also contractible. Show that X is contractible.

3. Consider the space X obtained by gluing $T^2 = S^1 \times S^1$ and $\mathbb{RP}^3$ by identifying the circle $\mathbb{RP}^1 \subset \mathbb{RP}^3$ with the circle $S^1 \times \{\text{point}\} \subset T^2$. Suppose that $g : X \to S^3$ is a continuous map such that $g|_{\mathbb{RP}^3}$ is homotopic to a constant map. Show that g is nullhomotopic.

4. Recall that S^∞ is a CW complex obtained as the union

$$S^0 \subset S^1 \subset S^2 \subset \cdots \subset S^n \subset S^{n+1} \ldots,$$

where the CW structures are chosen so that for each n, there are two n-cells: the n-sphere S^n is the equator of S^{n+1}, and two $(n+1)$-cells (top and bottom hemispheres) are attached to S^n to create S^{n+1}. (As always, S^0 consists of two points, with discrete topology.) The n-dimensional sphere S^n is the n-skeleton of S^∞.

Prove that S^∞ is contractible.

Hint: you will need to use the homotopy extension property. [Do not try to use question 2, it won't help.]

Note: it follows that all higher homotopy groups of $\mathbb{RP}^\infty$ vanish. (We already know that $\pi_1(\mathbb{RP}^\infty) = \mathbb{Z}/2$.)

5. Given a covering $p : \tilde{X} \to X$, define the *action of* $\pi_1(X, x_0)$ *on the fiber* $F = p^{-1}(x_0)$ as follows. Given $[\gamma] \in \pi_1(X, x_0)$, let $\phi_\gamma : F \to F$ be the map that sends x_1 to x_2 if the lift $\tilde{\gamma}$ starting at x_1 ends at x_2.

Explain why this is well-defined (i.e. ϕ_γ depends only on the homotopy class of γ), and why this gives an action in the following sense:

$$\phi_{\alpha\beta} = \phi_\beta \circ \phi_\alpha, \qquad [\alpha], [\beta] \in \pi_1(X, x_0),$$
$$\phi_e = id_F.$$

Note: there's this rather confusing business that comes up both here and especially when working with deck transformations: concatenation of loops and composition of homeomorphisms work in different order. (For this reason, we had to pass to some inverses when building the correspondence between $Deck(\tilde{X})$ and $\pi_1(X)$ which is an honest homomorphism rather than an "anti-homomorphism".) One can deal with this by introducing the formalism of "right actions" and "left actions" and anti-homomorphisms, which we tried to avoid by sticking to the consistent order of operations (at the cost of formulas that don't look so nice). I think Hatcher uses this formalism (without explaining it explicitly), so some of the conventions look different.

This question took me so long to put in because I wanted to make a connection with $Deck(\tilde{X})$ for the universal cover $\tilde{X}$ and build a covering X_H corresponding to a given subgroup of $\pi_1(X)$ as the quotient by the action of the corresponding subgroup of $Deck(\tilde{X})$. But this order of operations issue complicates the construction so much that it isn't useful. Sorry!