MAT 540, Homework 11, due Thursday, Nov 14

1. Let X be a (finite-dimensional) CW complex. Show that X is Hausdorff.

You will need to construct open neighborhoods inductively, working your way up the skeleta. Note that if U is an open set in X^{n-1}, then U is typically not open in X^n but can be enlarged to an open set in X^n.

2. Given a (finite-dimensional) CW complex X with skeleta

$$X^k = X^{k-1} \sqcup \bigsqcup_\alpha D^k_\alpha \,/(\phi_\alpha(x) \sim x, x \in \partial D^k_\alpha), k = 1, 2, \dots,$$

the *characteristic map* $\Phi_\alpha : D^k_\alpha \to X$ of the given cell is the composition

$$D^k_\alpha \hookrightarrow X^{k-1} \sqcup \bigsqcup_\alpha D^k_\alpha \to X^k \hookrightarrow X,$$

where the first and third maps are the obvious inclusions, and the second is the quotient map.

(a) Show that the restriction $\Phi_\alpha|_{\mathrm{Int}\, D^k_\alpha}$ to the interior of the disk is a homeomorphism onto its image. Common notation: $e^k_\alpha = \Phi_\alpha(\mathrm{Int}\, D^k_\alpha)$; we say that e^k_α is an *open cell.*

Show that e^k_α is open in X^k but that e^k_α may *not* be open in X if X has higher-dimensional cells.

We have a decomposition of X into a disjoint union of cells e^k_α of different dimensions, $X = \cup_{k,\alpha} e^k_\alpha$.

(b) Show that $\Phi_\alpha(D^k_\alpha)$ is closed in X, and that this is the closure of the open cell e^k_α. [You will need to use the Hausdorff property.] We say that $\Phi_\alpha(D^k_\alpha) = \bar{e}^k_\alpha$ is a *closed cell.*

3. Describe CW decompositions and compute the fundamental group for the following spaces (arguing from cell attachments).

(a) The quotient space X of S^2 under the identifications $x \sim -x$ for x in the equator S^1. (Note that X is *not* the same as $\mathbb{R}\mathrm{P}^2$: the opposite points outside the equator are *not* identified in pairs.)

(b) The CW complex Y obtained from S^1 by attaching two 2-cells, via maps $z \mapsto z^2$ and $z \mapsto z^3$, respectively. (Think of a 2-cell as unit disk $D \subset \mathbb{C}$, so that $\partial D = \{|z| = 1\}$).

(c) The space Z obtained from two tori $S^1 \times S^1$ by identifying a circle $S^1 \times \{x_0\}$ in one torus with the corresponding circle $S^1 \times \{x_0\}$ in the other torus.

(d) The wedge $\mathbb{R}\mathrm{P}^2 \vee T^2$ of the projective plane and the torus.

(e) The complex projective space $\mathbb{C}\mathrm{P}^n$.

For the CW structure on $\mathbb{C}\mathrm{P}^n$, note that it's not enough to say that $\mathbb{C}\mathrm{P}^k \setminus \mathbb{C}\mathrm{P}^{k-1}$ is homeomorphic to an open disk for each k: you need to have attaching maps for the cells to give a CW structure. It's not too hard to do this yourself but feel free to read it in Hatcher Chapter 0 if you get stuck.

4. This question asks you to prove a part of the statement $\pi_n(S^n, x_0) = \mathbb{Z}$, namely, the fact that every map from S^n to S^n is homotopic to a multiple of the identity map $\mathbf{1}$ (and the homotopy can be chosen to fix the basepoints). The strategy of the proof is in part (b).

(a) [Do not submit this part.] Make sure you understand the group operation in $\pi_n(S^n, x_0)$, what the multiples of $\mathbf{1}$ look like, what $-\mathbf{1}$ looks like, and why $\mathbf{1} + (-\mathbf{1}) = \mathbf{0}$. We discussed this material in class a few weeks ago, but please review as necessary.

(b) Given $f : (S^n, x_0) \to (S^n, x_0)$, homotop f to a smooth map g (rel x_0), and pick a regular value y of g, so that $g^{-1}(y) = \{x_1, \dots, x_k\}$. Use the result of Question 5 Homework 8 to change g by a homotopy in small neighborhoods of x_i's so that the map is the identity or the reflection (in the appropriate coordinates). Then make another homotopy to expand these special neighborhoods and stretch each of their images to the entire sphere to obtain a multiple of the identity map.

Note: The above argument shows that g is homotopic to $m \cdot \mathbf{1}$, where m is the number of preimages of a regular value of g, counted with sign. (The sign is given by the determinant of the Jacobian of the map at each x_i, after fixing the orientation of S^n: in other words, x_i is counted with $+$ or $-$ depending on whether the local diffeomorphism g preserves or reverses the orientation at x_i.) The number m is called the *degree* of g. It can be shown that m is independent of the choice of the regular value, that in fact $\deg g$ is a homotopy invariant. (See Milnor's Topology from the Differentiable Viewpoint.) Once you have the latter statement, it follows that maps of different degree cannot be homotopic, and you can conclude that $\pi_n(S^n, x_0) = \mathbb{Z}$. (The argument in part (b) shows that $\mathbb{Z}$ surjects onto $\pi_1(S^n, x_0)$.) Homotopy invariance of the degree can be established by expressing the degree via homology or cohomology (in MAT 531), or by an argument given in Milnor's book. [This note is for your information only.]

5. This question defines the winding number of a loop around a point and establishes its properties.

Suppose $u : S^1 \to \mathbb{R}^2$ is a continuous map, and $x \notin u(S^1)$. Then u determines an element $\operatorname{ind}_x u \in \pi_1(\mathbb{R}^2 - \{x\}) = \mathbb{Z}$, called **the winding number of u with respect to x**. [We fix the counterclockwise direction on S^1 which is the domain of u, and for each $x \in \mathbb{R}^2$, we will choose the homotopy class of the *couterclockwise* standard loop going once around x as the generator $1 \in \mathbb{Z}$.]

(a) Show that if $u(S^1)$ is contained in a disk D and $x \notin D$, then $\operatorname{ind}_x u = 0$.

(b) Prove that the formula $x \mapsto \operatorname{ind}_x u$ defines a locally constant function on $\mathbb{R}^2 - u(S^1)$. (It follows that if u is a "nice" curve, possibly with some self-intersections, so that it divides $\mathbb{R}^2$ into some connected components, then the winding number remains the same within each component.)

(c) Let $u : S^1 \to \mathbb{R}^2$, and suppose that $x, y \in \mathbb{R}^2 - u(S^1)$, such that $\operatorname{ind}_x u \neq \operatorname{ind}_y u$. Show that any path from x to y must intersect $u(S^1)$.

(d) Now assume that $u : S^1 \to \mathbb{R}^2$ is a smooth immersion, with a finite number of self-intersection points, $x \notin u(S^1)$ as before. Show that for almost all rays R in $\mathbb{R}^2$ starting at x, the ray R meets $u(S^1)$ transversely at finitely many points that avoid the self-intersections. [Note that $u(S^1)$ is locally a smooth submanifold in $\mathbb{R}^2$ away from the self-intersections. "Almost all rays" means the statement is true for all angles of the ray except for a set of measure 0.]

(e) Suppose that the ray R starting from x as above meets $u(S^1)$ transversely at finitely many points that avoid the self-intersections. Show that $\operatorname{ind}_x u$ equals to the signed count of intersections of R with the curve $u(S^1)$, where an intersection point p is counted with a $+$ if the curve traverses R in the counterclockwise direction at p, and with $-$ if the direction is clockwise. [More formally, you compare the orientation given by the outward ray and the tangent vector to the oriented curve with the standard orientation of $\mathbb{R}^2$.]

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The van Kampen theorem, which we did not discuss in class, is another useful tool for computing the fundamental group (although not always the best one). Please review that theorem if you are already familiar, or attend MAT 530 for a couple of lectures if you are not. You should be able to compute all the fundamental groups in Question 3 via van Kampen as well as via the cellular techniques.

Optional but recommended: Read sections 7.3 and 7.4 of Fomenko-Fuchs, Chapter 1 to learn about the knot group (the fundamental group of the complement of the knot) and the Wirtinger presentation; see also Hatcher Exercise 22 p.55, section 1.2. This is one situation where the van Kampen theorem is very hepful (it was actually developed for some related questions). Try to work through some of the exercises in Fomenko–Fuchs but do not submit anything.