MAT 540, Homework 10, due Thursday, Nov 7

We defined CW-spaces via an inductive procedure constructing the skeleta $X^0 \subset X^1 \subset X^2 \subset ....$ The 0-skeleton X^0 is a set of points with discrete topology. If the $(k-1)$-skeleton X^{k-1} is already constructed, we construct X^k as the quotient space obtained by attaching k-dimensional cells D^k_α (copies of the k-disk D^k) to X^{k-1} via attaching maps $\phi_\alpha : \partial D^k_\alpha \to X^{k-1}$, so that

$$X^k = X^{k-1} \sqcup \bigsqcup_\alpha D^k_\alpha \;/(\phi_\alpha(x) \sim x, x \in \partial D^k_\alpha).$$

For now, we assume that all our CW-complexes are finite-dimensional, so that the process terminates at a step n, and $X = X^n$. (In the infinite-dimensional case, we will need to specify topology on X.)

Working with CW-spaces requires a good understanding of quotient topology. Please review quotient topology if necessary.

1. Let X be the CW-complex with 0-cells $x_0, x_1, ...x_n, ...$ and 1-cells $I_1, I_2, \dots, I_n, \dots$, so that the 1-cell I_n has its endpoints attached to the 0-cells x_0 and x_n. Let Y be the subspace of $\mathbb{R}^2$ which is the union of closed intervals I_n, where I_n connects the origin and the point $z_n = e^{2\pi i/n}$, $n = 1, 2, \dots$. Show that X and Y are *not* homeomorphic.

2. Let X be a (finite-dimensional) CW space. Show that the following properties are equivalent:

(i) X is path connected;

(ii) X is connected;

(iii) The 1-skeleton X^1 is connected.

3. For coprime integers $p > q > 0$, let $L(p, q)$ be the quotient space of $S^3 = \{(z, w) \in \mathbb{C}^2 : |z|^2 + |w^2| = 1\}$ by the equivalence relation

$$(z, w) \sim (e^{2\pi i/p}\, z, e^{2\pi i q/p}\, w).$$

The space $L(p, q)$ is called a *lens space.*

(a) Show that the quotient map $S^3 \to L(p, q)$ is a covering map, and explain briefly why $L(p, q)$ is a smooth manifold.

(b) Compute the fundamental group $\pi_1(L(p, q))$ and show that $L(p, q)$ and $L(p', q')$ are not homotopy equivalent if $p' \neq p$.

4. Let M be a connected smooth manifold of dimension $m > 0$. Assume that M has no boundary (although everything works in the same way if ∂M is non-empty, so part (c) of this question is okay). As we had discussed in class, recall that an orientation of $\mathbb{R}^m$ is an equivalence class of an ordered basis, where the basis $(v_1, \dots, v_m)$ and the basis $(v'_1, \dots, v'_m)$ are equivalent if the matrix sending one of them to the other has positive determinant. Accordingly, for each $x \in M$ there are two choices of orientation $\mathcal{O}_x$ of T_xM. In a chart $U \subset M$ with coordinates $(x_1, \dots x_m)$, we have coordinates $(x_1, \dots, x_m, \frac{\partial}{\partial x_1}, \dots \frac{\partial}{\partial x_m})$ on TU, so that the bundle $TU \to U$ can be identified with $U \times \mathbb{R}^m \to U$. Fixing an orientation of $\mathbb{R}^m$, we then have a choice of orientation $\mathcal{O}_U$ on the chart U. An orientation on M is a choice of orientation of T_xM for every $x \in M$ which restricts to an orientation on every chart; M is called orientable if an orientation on M exists.

Since at every point of M we have two choices of orientations (with no preferred choice), a covering space naturally arises if we consider both choices.
Define

$$\tilde{M} = \{(x, \mathcal{O}_x) \mid x \in M, \mathcal{O}_x \text{ is an orientation of } T_xM\},$$

and let $p : \tilde{M} \to M$ be the projection, $p(x, \mathcal{O}_x) = x$.

We specify a basis of topology on $\tilde{M}$ to make M a topological space. If U is an open chart on M and $\mathcal{O}_U$ is a choice of orientation on U as above, let

$$\tilde{U}_{\mathcal{O}_U} = \{(x, \mathcal{O}_x) \in \tilde{M} \mid x \in U, \mathcal{O}_x \text{ orientation of } T_xM \text{ given by } \mathcal{O}_U\}.$$

(a) Check that the sets $\tilde{U}_{\mathcal{O}_U}$ give a basis of topology on $\tilde{M}$.
(b) Show that $p : \tilde{M} \to M$ is a 2-fold covering. Explain briefly why $\tilde{M}$ has a smooth structure, so that p is a local diffeomorphism between the smooth manifolds $\tilde{M}$ and M.
(c) Show that $\tilde{M}$ is orientable.
$\tilde{M}$ is called the *orientation covering* of M.
(d) If M is the Möbius band, then what is its orientation covering? Specify the space and the covering projection.
(e) Prove that M is orientable if and only if $\tilde{M}$ is a trivial 2-fold covering, that is, $\tilde{M}$ is given by two disjoint copies of M, and p restricts to a diffeomorphism on each copy. [M is assumed to be connected.] Conversely, it follows that M is non-orientable if and only if $\tilde{M}$ is connected.

5. A plane triangle is an unordered triple of points in $\mathbb{R}^2$ which are not collinear. Let T be the space of all plane triangles, and $R \subset T$ its subspace consisting of all right triangles. (What is the topology on T?)
(a) Find the fundamental groups of T and R. What is the homomorphism i_* induced by the inclusion $i : R \to T$?
(b) Does T retract to R?
There are some covering spaces lurking here.

6. (a) Prove that $SO(3)$ is homeomorphic to $\mathbb{R}\mathrm{P}^3$.
The easiest way to do this is probably to use the fact that because each 3×3 matrix has a real eigenvalue, each rotation of $\mathbb{R}^3$ has a fixed axis (thus, it is represented as a rotation about some axis by some angle).
(b) The group

$$SU(2) := \{A \in GL_2(\mathbb{C}) : \bar{A}^T A = I, \det A = 1\}$$

is the group of unitary 2×2 matrices of determinant 1, with matrix multiplication. (We consider matrices with complex coefficients, $\bar{A}$ stands for the matrix where we take the complex conjugate of each entry, T means the transpose.) As all linear groups, it is a topological space (a subset of $\mathbb{C}^4$, via matrix coefficients).
Show that elements of $SU(2)$ have the form

$$A = \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix},$$

where α and β are complex numbers, $|\alpha|^2 + |\beta|^2 = 1$, and that $SU(2)$ is homeomorphic to S^3.
Note: There is a group homomorphism $SU(2) \to SO(3)$ which is a 2-fold covering. To construct this homomorphism, consider the group of quaternions, which consists of eight elements $\pm\mathbf{1}, \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}$ with multiplication given by the rules $\mathbf{ij} = -\mathbf{ij} = \mathbf{k}$, etc. [] The space

$$\mathbb{H} = \{a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \mid a, b, c, d \in \mathbb{R}\}$$

is a 4-dimensional vector space with a (non-commutative) algebra structure (multiply quaternions by the usual rules and extend multiplication by linearity), with the standard norm where $\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}$ give an orthonormal basis. We can identify $SU(2)$ with the unit sphere in the space of quaternions $\mathbb{H}$.
If $V \subset \mathbb{H}$ is the linear subspace $V = \{b\mathbf{i} + \mathbf{cj} + \mathbf{dk}\}$, and $q \in SU(2)$ a unit quaternion, then

$$q(v) = qvq^{-1}, \quad v \in V$$

is an isometry of V, and therefore a rotation, which gives an element of $SO(3)$. This is a surjective 2:1 map (this can be checked either by a lengthy calculation or by a more conceptual argument). It is clear that the elements q and $-q$ define the same element of $SO(3)$, so it's the standard covering of $S^3 \to \mathbb{R}\mathrm{P}^3$.
[This note is for your information only. This covering is important in math and physics, so if you feel motivated, you can try to work out some of the details or look them up in the literature.]