CW complexes: a summary of what we covered

- Inductive definition of a CW complex X: start with a discrete space X^0. Assuming that X^{n-1} is already constructed, attach n-cells D^n_α to X^{n-1} via attaching maps $\phi_\alpha : \partial D^n_\alpha \to X^{n-1}$ to form the space X^n, with quotient topology. Set $X = \cup_{n\geq 1} X^n$, with weak topology. The space X^n is the n-skeleton of X.
- For each n-cell, we have a characteristic map $\Phi^n_\alpha : D^n_\alpha \to X$. Then X is the disjoint union of cells $e^n_\alpha = \Phi^n_\alpha(\operatorname{Int} D^n_\alpha)$. Each n-cell $e^n_\alpha \subset X$ is open in the corresponding skeleton X^n (but typically not open in X).
- Every CW complex is Hausdorff. We constructed open neighborhoods of points, working inductively with the skeleta X^n.
- From the Hausdorff property and compactness, it follows that $\Phi^n_\alpha(D^n_\alpha)$ is closed in X. Moreover, $\Phi^n_\alpha(D^n_\alpha) = \bar{e}^n_\alpha$ is the closure of the cell e^n_α.
- A subcomplex A of a CW complex X is the union of some collection of cells e^m_β in X, such that the closure of each cell is contained in A. A subcomplex is a closed subset of X. A subcomplex is a CW complex, and the CW structure (given by the cells of X) gives the same topology as the subspace topology on $A \subset X$.
- Every finite CW complex is compact. Every compact subset C of a CW complex X is contained in a finite subcomplex $A \subset X$ (in particular, C can only intersect finitely many cells).

The proofs of these statements can be found, for example, in Appendix to Hatcher or other books.
I think it's best to go in the order as above: it's easy to get caught up in circular logic.

Note: The classical definition of CW complexes, due to Whitehead, doesn't use the inductive approach, instead describing X as a Hasdorff space given as the union of cells, with topology satisfying certain axioms (C) and (W). This is the definition the Fomenko–Fuchs book uses (but they leave most of the properties as an exercise). Recommended reading: Proposition A.2 and its preceding discussion in Hatcher Appendix shows equivalence of the two definitions. (We didn't cover this in class because the inductive definition is typically easier to work with; the point is that you can build arguments working cell-by-cell and using induction on dimension and weak topology.)

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- A function $f : X \to Y$ is continuous iff its restriction to each skeleton is continuous. A homotopy $F : X \times I \to Y$ is continuous iff its restrictions to each $X^n \times I$ is continuous. (The statement about the homotopy is non-trivial and follows from the fact that the weak topology on $X \times I$ induced by the filtration by $X^n \times I$'s is equivalent to the product topology. See lemma below.)
- Homotopy extension property for CW pairs: we followed the proof from Fomenko–Fuchs to construct the extension directly cell-by-cell (continuity of the resulting homotopy follows from the statement above).
- Corollary: if (X, A) is a CW pair, and A is contractible, then $X/A \sim X$.

Lemma. *Let X be a CW complex. Suppose that $U \subset X \times I$ is a subset such that $U \cap (X^n \times I)$ is open for each n. Then U is open in the product topology on $X \times I$.*

Proof. Let $(x_0, s_0) \in U$, then we can find an open neighbohood $(a, b) \ni s_0$ such that $\{x_0\} \times [a, b] \subset U$. Set

$$V = \{y \in X | \, \{y\} \times [a, b] \subset U\}.$$

Then $V \cap X^n$ is open in X^n for each n. Indeed, we can use the "tube lemma". Fix an arbitrary $y \in V \cap X^n$. Then, since U is open in $X^n \times I$, for every $t \in [a, b]$ there is an open neighborhood V_t, $y \in V_t \subset X^n$ and $\epsilon_t > 0$ such that $V_t \times (t - \epsilon_t, t + \epsilon_t) \subset U$. The intervals $(t - \epsilon_t, t + \epsilon_t)$ cover $[a, b]$; choosing a finite open subcover corresponding to some $t_1, t_2, \ldots, t_k$, we see that

$$y \in \bigcap_{i=1}^{k} V_{t_k} \subset V \cap X^n,$$

so every y has an open neighborhood, and $V \cap X^n$ is open. Then V is open in X, and it follows that $V \cap (a, b)$ is a product neighborhood of (x_0, s_0) in $X \times I$. □

A similar argument can be used to show that the weak topology coincides with the product topology of two CW spaces X and Y if X or Y is locally compact (but in general, the two topologies on $X \times Y$ can be different).