MAT 539 Algebraic Topology

Problem Set 5

due Monday, December 13 (you can hand it in anytime on or before Dec 13)

Hatcher 3, 7 p.229 (the ring structure of $H^*(\mathbb{R}\mathrm{P}^n)$ and $H^*(\mathbb{C}\mathrm{P}^n)$ will be determined in Monday class).

***

Problem 1. Show that if X is a finite simplicial complex whose underlying topological space is a homology n-manifold, then

(a) X consists entirely of n-simplices and their faces,

(b) Every $(n-1)$-simplex is a face of precisely two n-simplices.

Problem 2. Suppose that X is a compact triangulable homology n-manifold.

(a) Show that if X is orientable, then $H_{n-1}(X)$ is torsion-free, and for any coefficient group G, $H_n(X;G) = G = H^n(X;G)$.

(b) Show that if X is non-orientable, then the torsion subgroup of $H_{n-1}(X)$ is $\mathbb{Z}/2$, $H_n(X;G) = \ker(G \to^2 G)$, and $H^n(X;G) = G/2G$.
In particular, $H_n(X) = 0$, $H^n(X) = \mathbb{Z}/2$.

Problem 3. Let X be a homology n-manifold (not necessarily compact) that is triagulated by a locally finite simplicial complex. ("Locally finite" means that each vertex belongs to finitely many simplices.) The Poincaré duality holds in this setting if one considers *cohomology with compact support* $H^p_c(X;G)$. To define this for a simplicial complex K, let $C^p_c(K;G) \subset \mathrm{Hom}(C_p(K),G)$ be the group of homomorphisms that vanish on all but finitely many oriented simplices of K.

(a) Show that if K is locally finite, then δ maps C^p_c into C^{p+1}_c, thus the resulting cohomology groups $H^p_c(K;G)$ are well defined.

(b) If K is the complex whose total space is $\mathbb{R}$ and whose vertices are the integers, show that $H^1_c(K) = \mathbb{Z}$ and $H^0_c(K) = 0$.

(c) Show that $H^p_c(X;\mathbb{Z}/2) = H_{n-p}(X;\mathbb{Z}/2)$.

(d) We say that X is orientable if it's possible to orient the n-simplices σ_i of X such that the resulting (possibly infinite) chain $\sum \sigma_i$ is a cycle (i.e. $\partial(\sum \sigma_i) = 0$ when computed formally). Show that in this case $H^p_c(X;G) = H_{n-p}(X;G)$.

Problem 4. Let X, Y be compact connected triangulable homology n-manifolds.

(a) Show that if X and Y are orientable, and there is a continuous map $f : X \to Y$ has non-zero degree, then $b_i(X) \geq b_i(Y)$, where b_i stands for the i-th Betti number.

(b) Denote by X_g the orientable closed surface of genus g. Show that there exists a continuous map $f : X_n \to X_m$ of non-zero degree if and only if $n \geq m$.

Problem 5. Show that the Euler characteristic of a compact orientable triangulable homology manifold of odd dimension is always zero.