MAT 530, Homework 13, due Wednesday, 12/2

Questions 1-8 provide a review of the algebraic topology part of the course for the final exam and the comps. Questions 9-10 are not mandatory; I recommend that you think about them, but you don't have to submit solutions. (Complete solutions of these two might be a bit tedious to write down.)

1. Hatcher, question 21 p.81.

2. (a) Find a group G acting on the torus T (decribe the action) so that T/G is homeomorphic to T.

(b) Find a group G acting on the torus T (decribe the action) so that T/G is homeomorphic to the Klein bottle K. (Thus, you get a covering $T \to K$.)

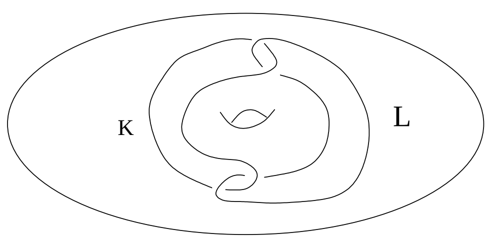
(c) Prove that there is no covering $K \to T$.

3. (a) Show that any 2-fold covering has a non-trivial deck transformation.

(b) Give an example of a 3-fold covering without any non-trivial deck transformations.
(Assume that all spaces are "nice" - path-connected, locally path-connected.)

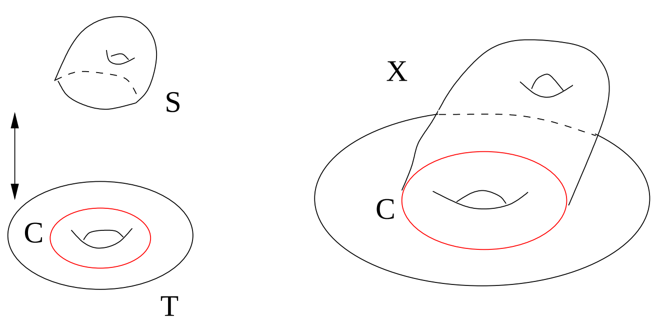
4. Let S be the solid torus $S = D^2 \times S^1$, and K, L two simple closed curves in S, as shown in the picture. (Everything is a subset of $\mathbb{R}^3$.) Notice that if you are allowed to move K and L outside the solid torus (over the hole), then these two curves can be untangled from one another (visual intuition suggests that K and L cannot be untangled inside S).

Prove that K is not null-homotopic in the complement of L in S. (Think of K as the image of a loop $\gamma : S^1 \to S \setminus L$; you need to show that there is no homotopy $F : S^1 \times I \to S \setminus L$ such that $F(t, 0) = \gamma(t)$ and $F(t, 1) = \text{const.}$)



5. Let X be a path-connected, locally path-connected space with a finite fundamental group. Show that any map $f : X \to S^1$ is null-homotopic.

6. Let T be a torus and C a homotopically non-trivial, simple closed curve in T. Let S be a torus with an open disk removed. Let X be a space obtained by gluing S to T so that the boundary ∂S of the surface S is identified with C.



(a) Write a presentation for $\pi_1(X)$.

(b) Show that C must lift to a closed loop (as opposed to an open path) on any 2-fold cover of X.

7. Let S^n be the unit sphere in $\mathbb{R}^{n+1}$ and let s_0 be the point $(1, 0, \dots, 0) \in S^n$. Define $\pi_n(X, x_0)$ to be the set of homotopy classes of maps $f : (S^n, s_0) \to (X, x_0)$. (We assume that homotopies fix the basepoint.)

Let $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be a covering. Define the map $p_* : \pi_n(\tilde{X}, \tilde{x}_0) \to \pi_n(X, x_0)$ by $p_*([f]) = [p \circ f]$.
(a) Show that p_* is well-defined.
(b) Show that for $n \geq 2$, p_* is a bijection.

8. Think of S^3 as $\{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$. Fix two relatively prime positive integers n, k. Consider the action of the cyclic group $\mathbb{Z}_n$ of order n on S^3, such that the generator h of $\mathbb{Z}_n$ acts by

$$h(z_1, z_2) = (z_1 e^{2\pi i/n}, z_2 e^{2\pi i k/n}).$$

(Check that h^n is the identity map.) The quotient space of this action, $S^3/\mathbb{Z}_n$, is called the lens space $L(n, k)$.

Show that if $L(n, k)$ and $L(n', k')$ are homeomorphic, then $n = n'$.

Optional questions, do not submit:
9. Use topology to show that any subgroup of a free group is free. (To avoid technicalities but still see the idea, you can make extra assumptions: the group is finitely generated, the subgroup has a finite index, etc.)

10. Show that any finite cellular space embeds into a Euclidean space of sufficiently high dimension. (Use 42.N.2 in Viro's book.)