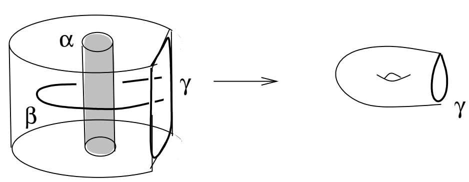
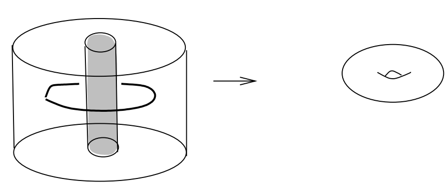
SOME SOLUTIONS TO HOMEWORK PROBLEMS

Hatcher 9 p.53 Solution Suppose that $r : M_h' \to C$ is a retraction; let $i : C \to M_h'$ be inclusion. We know that if retraction exists, then the map $i_* : \pi_1(C) \to \pi_1(M_h')$ is injective. As Hatcher suggests, abelianize π_1, and consider the map $i_*^{ab} : \pi_1(C) \to \pi_1(M_h')/[\pi_1(M_h'), \pi_1(M_h')]$. Notice that this works because the commutators go to commutators, and $\pi_1(C) = \mathbb{Z}$ is already abelian. But we know that if the surface M_h is represented by a polygon with sides $a_1, b_1, a_1, b_1, a_2, b_2, ...$ identified as usual, and the hole in M_h' is cut in the center of the polygon, then the loop C is given by the expression $a_1 b_1 a_1^{-1} b_1^{-1} ...$ which becomes trivial after taking the commutators. Thus i_*^{ab} is trivial, a contradiction. To construct the retraction to the circle C', first consider the case when the whole surface is the torus, and project to C'; for the general case, we can first map the surface to the torus.

Hatcher 10 p.53 Solution The complement of $\alpha \cup \beta$ on the figure from Hatcher is homeomorphic to the gadget shown on top left picture, and deformation retracts to a torus with a hole on top right. The curve γ goes to the boundary of the hole, and is not null-homotopic. (Recall that torus with a hole deformation retracts to a figure 8 whose fundamental group is the free group on two generators, a and b; the boundary of the hole gives a non-trivial loop $aba^{-1}b^{-1}$.)





Munkres 9e, §58: show that the maps $f_1, f_2 : S^1 \to S^1$ that have the same degree are homotopic.
Solution. We assume that S^1 is the unit circle in $\mathbb{C}$, $x_0 = 1$ (by (a) we can choose an arbitrary basepoint).
The statement we are trying to prove is similar to the fact that loops based at the same point and winding around S^1 the same number of times are homotopic

(this is because $\pi_1(S^1) = \mathbb{Z}$); but an additional difficulty is that perhaps $f_1(x_0) \neq f_2(x_0)$.

We first show that we can assume that $f_1(x_0) = f_2(x_0) = x_0$: otherwise instead of f_j consider the function $r \circ f_j$, where r is the rotation of the circle that sends $f_j(x_0)$ to x_0. Notice that any rotation is homotopic to the identity (via rotating by smaller angles), so $r \circ f_j$ and f_j are homotopic (and thus have the same degrees).

Now, we can think of the maps f_1, f_2 as loops in S^1 based at x_0, by considering the loops $\alpha_j(t) : [0,1] \to S^1$ given by $\alpha_j(t) = f_j(e^{2\pi it})$, $j = 1, 2$. Since the loop $t \mapsto e^{2\pi it}$ gives the generator of $\pi_1(S^1, x_0)$ denoted in Munkres by $\gamma(x_0)$, if $\deg f_1 = \deg f_2 = g$, then the loops α_1, α_2 both give the element $d \in \pi_1(S^1, x_0)$. It means that α_1, α_2 are homotopic (as loops), via a homotopy $F(t,s)$. We can go back, and consider each $\alpha_s(t) = F(t,s)$ as a map f_s from $S^1 \to S^1$ such that $\alpha_s(t) = f_s(e^{2\pi it})$. Then f_1, f_2 are homotopic thru maps f_s.

Question 2 hw 10: Prove that the complement of two linked loops in $\mathbb{R}^3$ and the complement fo two unlinked loops in $\mathbb{R}^3$ are not homeomorphic, by showing that they have different fundamental groups.

Solution. By stretching out the removed loops, we can show that the complement of two unlinked loops deformation retracts to $S^1 \vee S^2 \vee S^1 \vee S^2$, and the complement of the linked loops def. retracts to $S^2 \vee T^2$, where T^2 is the two-torus. To see this, first figure out why the complement of one circle def. retracts to $S^2 \vee S^1$. In fact, it may be easier to work with complements of these loops in S^3: $S^3 -$ circle $= \mathbb{R}^3 -$ line def retracts to a circle, $S^3 -$ linked loops def retracts to the two-torus. I may post some pics tomorrow if I have the time.