MAT 530 Topology, Geometry I

Problem Set 1

due Friday, September 11

Problem 1. Show that every open set in the standard topology on $\mathbb{R}$ is a *countable disjoint* union of open intervals.

Problem 2. Metrics d_1 and d_2 on a set X are called *equivalent* if there are constants $C, c > 0$ such that

$$cd_1(\mathbf{x}, \mathbf{y}) \leq d_2(\mathbf{x}, \mathbf{y}) \leq Cd_1(\mathbf{x}, \mathbf{y})$$

for all $x, y \in X$. Show that equivalent metrics induce the same topology on X.

Problem 3. Let C be the space of real-valued continuous functions on $[0, 1]$. For $f, g \in C$ define

$$d_1(f, g) = \int_0^1 |f(x) - g(x)| dx, \qquad d_{sup}(f, g) = \sup_{0 \leq x \leq 1} |f(x) - g(x)|.$$

Show that d_1 and d_{sup} are metrics on C.

Prove that the topologies induced on C by these metrics are *different*. Is it true that one of them is finer than the other?

Problem 4. Let (X, d) be a metric space. Consider $Y = X \cup \{a\}$, where $a \notin X$ (i.e. Y is X with an extra point added). Define collections $\mathcal{T}_1$, $\mathcal{T}_2$ of subsets of Y as follows:

$U \in \mathcal{T}_1$ iff either $U \subset X$ and U is open in (X, d), or $U = Y$.

$U \in \mathcal{T}_2$ iff either $U = \emptyset$, or $U = V \cup \{a\}$, where $V \subset X$ is open in (X, d).

Check whether each of $\mathcal{T}_1$, $\mathcal{T}_2$ is a topology on Y, and if so, whether it can be induced by any metric on Y.

Problem 5. Let $\mathcal{B}$ be a basis for the standard topology on $\mathbb{R}$. Prove that $\mathcal{B}$ can always be decreased, i.e there is a set $U \in \mathcal{B}$ such that $\mathcal{B} - \{U\}$ is still a basis for the standard topology.

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Please also do questions 1, 5 (basis part only) and 8b of Munkres §13.