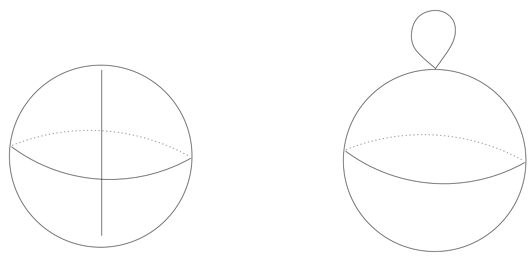
MAT 530 Topology, Geometry I

Problem Set 11

due Friday, December 4

1. Prove that the spaces shown on the picture (the union of sphere and its diameter, and the wedge sum of a sphere and a circle) are homotopy equivalent. Although formulas are not required, please describe the homotopy equivalences and the relevant homotopies clearly.



Hatcher p.53, Questions 9, 10 and 14.

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We were proving the general lifting lemma in class (Hatcher, Proposition 1.33). We constructed the lift and will finish the proof next Monday. In the meanwhile, you can use this lemma (as well as the uniqueness statement of Hatcher, Proposition 1.34) for the following questions.

Hatcher p. 79, Questions 7 and 9.

2. Prove that any continuous map f from the sphere S^2 to the torus T is null-homotopic.

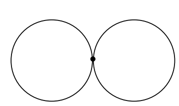
3. (a) Suppose $p_1 : \tilde{X}_1 \to X$, $p_2 : \tilde{X}_2 \to X$ are coverings, $\tilde{X}_1, \tilde{X}_2, X$ path-connected, locally path connected. Suppose that $p_*\pi_1(\tilde{X}, \tilde{x}_1) \subset p_*\pi_1(\tilde{X}_2, \tilde{x}_2)$. Show that there is a continuous map $\phi : X_1 \to X_2$, compatible with the coverings so that $p_1 = p_2\phi$.

(b) Prove that the map ϕ you found above is a covering map.

(This proves the Hierarchy of Coverings statement we discussed in class.)

4. Let $p : \tilde{X} \to X$ be a covering, $\tilde{X}$ path-connected space. Consider f a loop in X based at x_0; we can lift f to paths in $\tilde{X}$ starting at different points in $p^{-1}(x_0) \subset X$. It turns out that we may get one lift which is a closed loop, and another which is not.

(a) Let X be a figure 8 space (see picture). Give an example of a 3-fold covering $p : \tilde{X} \to X$, a loop f based at $x_0 \in X$, and two points $\tilde{x}_1, \tilde{x}_2 \in p^{-1}(x_0)$, such that a lift of f staring at $\tilde{x}_1$ is a closed loop, and a lift starting at $\tilde{x}_2$ is not.



(b) show that the following two conditions are equivalent:

(1) For every loop f based at $x_0 \in X$, either all lifts of f to $\tilde{X}$ are closed loops, or none of the lifts are closed (regardless of the starting points of the lifts).

(2) $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ is a normal subgroup of $\pi_1(X, x_0)$ for some $\tilde{x}_0 \in p^{-1}(x_0)$

Hint: what is the relation between the subgroups $p_*(\pi_1(\tilde{X}, x_1))$ and $p_*(\pi_1(\tilde{X}, \tilde{x}_2))$ for different $\tilde{x}_1, \tilde{x}_2 \in p^{-1}(x_0)$?

(c) If the conditions of (b) are satisfied, $p : \tilde{X} \to X$ is called a normal covering. Check that this notion is independent of the choice of $x_0 \in X$.

(d) Show that every two-fold covering is normal.

5. In a previous homework, we considered the quotient space X of the sphere S^2 obtained by identifying the north pole with the south pole. We computed $\pi_1(X) = \mathbb{Z}$. Describe explicitly (with proofs) the universal covering of X and all 5-fold coverings of X.

Read in Hatcher (pp.57–59) the discussion about the coverings of the figure 8 space. Pay attention to the following algebraic corollary: for every n, the free group on 2 generators contains a subgroup isomorphic the free group on n generators. This is because we can construct a covering space $(\tilde{X}, \tilde{x}_0)$ whose fundamental group is a free group on n generators, and we know that $p_* : \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$ is a monomorphism.

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