MAT 364, Homework 9, due 11/3

1. (a) Prove that in a metric space (X, d), every closed ball

$$\bar{B}_r(x) = \{y : d(x, y) \leq r\}$$

is indeed a closed set in metric topology on X.

(b) Is it always true that $\bar{B}_r(x)$ is the closure of the open ball $B_r(x)$, and that $B_r(x)$ is the interior of $\bar{B}_r(x)$? Prove or give counterexamples.

2. Determine (with proof) whether the following spaces are Hausdorff:

(a) $X = \{a, b, c, d\}$, with topology given by basis $\{\{a\}, \{b\}, \{c, d\}, \{c\}\}$ (no need to check that it's a basis).

(b) the digital line topology (Homework 6 Question 2)

3. (a) Prove that in a Hausdorff space X, points are closed, that is, $\{x\}$ is closed for every $x \in X$.

(b) Give an example of a space Y where every point is closed but Y is not Hausdorff.

4. Let X be a topological space, $A \subset X$.

(a) If X is Hausdorff, does it follow that A must be Hausdorff (with subspace topology)? Prove or give a counterexample.

(b) Can a non-Hausdorff space X have subset A (containing at least 2 points) which is Hausdorff? Give an example (with justification) or show that this is not possible.

5. Does there exist a continuous surjective function

(a) $f : S^1 \to (0, 1)$?

(b) $g : (0, 1) \to S^1$?

As usual, S^1 denotes a circle, with its standard topology.

6. Let (X, d) be a metric space, $a \in X$.

(a) Consider the function $f : X \to \mathbb{R}$ defined by $f(x) = d(a, x)$. Show that f is continuous. We are assuming that X has the metric topology, $\mathbb{R}$ has its standard topology. It will be easier to argue with the ϵ-δ-definition.

(b) Let $A \subset X$ be a non-empty compact set, $a \notin A$. Show that $\inf_{x \in A} d(a, x) > 0$, and there is a point $x_0 \in A$ such that

$$d(a, x_0) = \inf_{x \in A} d(a, x).$$

This quantity is called the distance between the point a and the set A.

Hint: we know a useful theorem about continuous functions.

(c) Give an example (with justification) of two non-empty disjoint closed sets A, B in $\mathbb{R}^2$ with the standard Euclidean metric, such that $\inf_{x \in A, y \in B} d(x, y) = 0$.

Optional: if A, B are two non-empty disjoint compact sets in a metric space, upgrade the argument from (b) to show that $\inf_{x \in A, y \in B} d(x, y) > 0$, and there exist $x_0 \in A, y_0 \in B$ such that

$$\inf_{x \in A, y \in B} d(x, y) = d(x_0, y_0)$$

.