MAT 364, Homework 8, due 10/27

1. Consider $\mathbb{R}$ with the topology of finite complements. Is this space compact? Prove your answer.

2. Consider the upper limit topology on $\mathbb{R}$, given by the basis of half-open intervals of the form $(a, b]$, $a, b \in \mathbb{R}$. Is the closed interval $[0, 1] \subset \mathbb{R}$ compact in the corresponding supspace topology? Prove your answer.

3. Use compactness to prove that

(a) an open interval is not homeomorphic to a closed interval (we already know another proof);

(b) the sphere S^n is not homeomorphic to $\mathbb{R}^n$, for any n.

(c) What's wrong with this argument:

Let A, B be two subsets of $\mathbb{R}^n$ such that A is bounded, B is not bounded. Then A cannot be homeomorphic to B because A is compact but B is non-compact.

4. (a) Suppose that X is compact, $V_1 \supset V_2 \supset V_3 \supset \cdots \supset V_n \supset \ldots$ are non-empty closed subsets of X. Prove that the intersection $\cap_{i=0}^{\infty} V_i$ is non-empty. (This generalizes the nested closed intervals property for $\mathbb{R}$.)

(b) Suppose that X is a compact topological space. Let $\{V_\alpha\}$ be a collection of closed subsets of X, such that the intersection of any *finite* subcollection $V_{\alpha_1}, V_{\alpha_2}, \ldots, V_{\alpha_k}$ is non-empty:

$$V_{\alpha_1} \cap V_{\alpha_2} \cap V_{\alpha_k} \neq \emptyset.$$

Show that then, the intersection of *all* of the sets V_α must be non-empty, $\cap_\alpha V_\alpha \neq \emptyset$.

(c) Conversely, suppose that the above property holds for every collection of closed sets: if the intersection of each finite subcollection is non-empty, then the intersection of all sets in the collection is non-empty. Prove that X is compact.

(d) Give an example of a non-compact space X with a sequence of closed subsets $V_1 \supset V_2 \supset V_3 \supset \cdots \supset V_n \supset \ldots$ such that the intersection $\cap_{i=0}^{\infty} V_i$ is empty.

5. (a) Let $A \subset \mathbb{R}^n$ be a subset which is closed and bounded, $f : A \to \mathbb{R}^n$ is a continuous function. Show that $f(A)$ is closed and bounded.

(b) Give an example of a closed subset $A \subset \mathbb{R}$ and a continuous function $f : A \to \mathbb{R}$ such that $f(A)$ is not closed.

(c) Give an example of a bounded subset $A \subset \mathbb{R}$ and a continuous function $f : A \to \mathbb{R}$ such that $f(A)$ is not bounded.

Optional: a product of compact spaces is compact. No points.

If you did the optional assignment on product topology, read about the product of compact spaces (Lemma 7.9, Theorem 7.10 in the textbook); do exercises 7.10 and 7.12.

Optional: compactness in metric spaces. No points.

If you know what a metric space is, prove that a compact subset in a metric space must be closed and bounded. The topology on a metric space is given by the basis consisting of open balls $B_r(x) = \{y : d(x, y) < r)\}$. Your proof should be very similar to the proof in $\mathbb{R}^n$.

Give an example of a metric space with a closed and bounded subset which is not compact.