MAT 364, Homework 7, due 10/20

In Questions 1-3, use connectedness as a tool. You may use the facts we proved in class (such as connectedness of intervals) but please prove everything else; do not use any statements from the textbook that we didn't discuss.

1. Prove that $[a, b]$ is not homeomorphic to (a, b).

2. Prove that $\mathbb{R}$ is not homeomorphic to any $\mathbb{R}^n$ for $n \geq 2$.

Note: There exist continuous surjective functions $\mathbb{R} \to \mathbb{R}^2$ and $[0, 1] \to [0, 1] \times [0, 1]$, even though this contradicts our intuition. If you are interested, read about "Peano curves" aka "space filling curves" somewhere.

3. Prove that the circle S^1 is not homeomorphic to any sphere S^n of dimension $n \geq 2$.

4. Prove a version of the intermediate value theorem for continuous functions $f : X \to \mathbb{Z}$, where $\mathbb{Z}$ has topology $\mathcal{T}_d$ defined in Homework 6 Question 2, and X is a connected topological space.

The space $(\mathbb{Z}, \mathcal{T}_d)$ is called *the digital line*: it mimics the topological properties of $\mathbb{R}$ but has only countably many points.

5. Do question 6.32 from the Adams–Francosa textbook.

Optional but Recommended: Path connected spaces. No points.

Let X be a topological space, $x, y \in X$. A *path* in X from x to y is a continuous function $\gamma : [0, 1] \to X$ with $\gamma(0) = x$, $\gamma(1) = y$. (Think of it as a trajectory of a point traveling in X from time $t = 0$ to $t = 1$.)

X is called *path connected* if for every pair of points $x, y \in X$ there is a path from x to y.

Prove that if X is path connected, then X is connected. Hint: use the fact that $[0, 1]$ is connected.

This is an important theorem (read the proof in the textbook if you get stuck). It gives a useful way to check that a space is connected (paths are often easy to construct). Note that the converse is not true: there are connected spaces that are not path connected. (Examples are tricky; see textbook.)

(a) Prove that every convex subset of $\mathbb{R}^n$ is path connected.

(b) Prove that $\mathbb{R}^n \setminus C$, where C is a countable set, is path connected.

(c) Prove that the digital line is path connected.

It follows that all of the spaces above are connected (you could also check connectedness directly).

Prove that if $f : X \to Y$ is continuous, and X is path connected, then $f(X)$ is path connected. Corollary: if X and Y are homeomorphic, then they are either both path connected or both are not.