MAT 364, Homework 6, due 10/13

1. Let X be the set $\mathbb{R}$ with the finite complements topology $\mathcal{T}$. (A non-empty set is open iff its complement is finite; the empty set is also open.)

(a) Is $(X, \mathcal{T})$ connected? Prove your answer.

(b) Does $(X, \mathcal{T})$ have any subsets that are disconnected? Give an example or show that disconnected subsets don't exist.

(c) Recall that the collection of half-open intervals of the form $(a, b]$, $a, b \in \mathbb{R}$, forms a basis of a topology on $\mathbb{R}$ called *upper limit topology.* (No need to check that this is a basis, see Homework 3, question 1-iii). Is $\mathbb{R}$ connected with the upper limit topology?

2. Let $\mathbb{Z}$ be the set of integers, and consider the collection $\mathcal{B}$ that consists of all subsets of the form $\{2n\}$ (one-point subsets given by the even integers) as well as all subsets of the form $\{2n, 2n+1, 2n+2\}$, $n \in \mathbb{Z}$.

(a) Sketch these subsets, and check that they form a basis of a topology on $\mathbb{Z}$. Let $\mathcal{T}_d$ denote this topology.

(b) Show that $(\mathbb{Z}, \mathcal{T}_d)$ is *not* homeomorphic to $\mathbb{Z}$ with discrete topology, nor to $\mathbb{Z}$ with finite complements topology.

(c) Prove that $(\mathbb{Z}, \mathcal{T}_d)$ is connected.

3. (a) Prove that $[0, 1)$ is homeomorphic to $[0, +\infty)$, by providing an explicit homeomorphism. (You may use continuous functions from calculus).

(b) Consider $\mathbb{R}$ with two different topologies, the upper limit topology generated by the basis $\{(a, b]\}_{a,b\in\mathbb{R}}$, and the lower limit topology generated by the basis $\{[a, b)\}_{a,b\in\mathbb{R}}$. Are these two topological spaces homeomorphic? Justify your answer.

4. (a) Suppose that X is disconnected, and $X = U \cup V$ is a separation of X. (That is, U and V are disjoint non-empty open subsets of X.) Let A be a *connected* subset of X. Show that $A \subset U$ or $A \subset V$. Argue from definitions.

(b) Let $A_1, A_2, \dots A_k, \dots$ be connected subsets of a topological space Y. Suppose that $A_k \cap A_{k+1} \neq \emptyset$ for every $k \in \mathbb{N}$. Show that the union $\bigcup_{k=1}^{\infty} A_k$ is connected. (Argue by contradiction and use part (a) and induction).

5. Let X be a topological space, $A \subset X$. Prove that $Cl\, A$ is connected if A is connected.

Argue from the definition of the closure and any of the equivalent characterizations of a connected subspace.

Optional Topic: Product Topology Strongly recommended for students who are comfortable with the class material. No points will be given.

(a) Let X, Y be topological spaces. Consider the collection of subsets $U \times V \subset X \times Y$, where U is open in X, V is open in Y. Check that this collection forms a basis for some topology on the set $X \times Y$. This topology is called *product topology*, and $X \times Y$ is called the product of topological spaces X and Y.

(b) Show that the projection maps $p_X : X \times Y \to X$ and $p_X : X \times Y \to Y$, defined by $p_X(x, y) = x$ and $p_Y(x, y) = y$, are continuous maps.

(c) Prove that if X and Y are connected, then $X \times Y$ is connected.

If you get stuck, read the proofs in the textbook. Parts (a) and (b) are easy exercises, (c) is harder.