MAT 364, Homework 5, due 9/29

In questions 1, 2, and 5, $\mathbb{R}$ is always considered with its standard topology.

1. Consider $A = (0,3] \cup \{5\} \subset \mathbb{R}$, with the subspace topology. For each of the following subsets of A, decide whether it is open and whether it is closed. (Remember that sets can be both open and closed, or neither open nor closed.) Justify your answers.

(a) $(2,3)$ (b) $(0,3]$ (c) $\{2\}$ (d) $(2,3]$ (e) $\{5\}$

2. (a) Consider

$$Y = \left\{\frac{1}{n},\ n \in \mathbb{N}\right\} = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right\} \subset \mathbb{R}$$

with the subspace topology. Show that this topology is the same as the discrete topology on Y.

(b) Let $Z = Y \cup \{0\}$, with the subspace topology. Is this topology the same as the discrete topology on Z?

Note: due to an inconsistency in the previously posted version of the homework, solutions that work with the set $Y' = \left\{\frac{1}{n},\ n \in \mathbb{Z}\right\}$ instead of Y will also be accepted.

3. Suppose X is a topological space, $Y \subset X$ has subspace topology.

(a) Show that if Y is open (in X), and A is open with respect to the subspace topology on Y, then A is open in X.

(b) Show that if Y is closed (in X), and B is closed with respect to the subspace topology on Y, then B is closed in X.

4. Let X, Y be topological spaces, $f : X \to Y$ a continuous function.

(a) Suppose that A is a subset of X, and define the function $g : A \to Y$ by $g(x) = f(x)$ for all $x \in A$. (Such g is called a "restriction" of f; intuitively, the function is exactly the same, except it is defined on a smaller domain.) Show that g is continuous.

(b) Let $Z = f(X) = \{y \in Y : y = f(x) \text{ for some } x \in X\}$. Note that $Z \subsetneq Y$ if f is not surjective.

Define the function $h : X \to Z$ by $h(x) = f(x)$. (Functions f and g are essentially the same, but h has a smaller target space.) Show that h is continuous.

5. Let $f : (-\infty, 0] \to \mathbb{R}$ and $g : [0, +\infty) \to \mathbb{R}$ be two continuous functions such that $f(0) = g(0) = c$. Let

$$h(x) = \begin{cases} f(x) & \text{if } x < 0 \\ g(x) & \text{if } x \geq 0. \end{cases}$$

Prove that $h : \mathbb{R} \to \mathbb{R}$ is continuous.

Please work directly from the topological definition of continuity. It may be easier to use the version with closed sets: a function is continuous if and only if the preimage of every closed set is closed. (Solutions using arguments from analysis, sequences, etc will receive no credit.)

6. Let $A = \{(x, y) \in \mathbb{R}^2 : x = 0\}$ be the x-axis in $\mathbb{R}^2$. The set A has two topologies: the subspace topology as a subset of $\mathbb{R}^2$, and the standard topology on $\mathbb{R}$ if you think of the x-axis as the standard real line. Compare these two topologies. (It will be useful to think about the basis; use Theorem 3.5.)

7*. (Optional challenge; do it if you can, no points will be given.)

Let X be a topological space, $A \subset X$. Prove that the subspace topology on A is the coarsest topology in which the inclusion $i : A \to X$ is continuous. (The inclusion map is defined by $i(x) = x$ for $x \in A$.)