MAT 364, Homework 4 due 9/22

Many questions in this assignment have multiple parts and/or if-and-only-if statements. Please make sure you don't miss any of the parts.

Several questions ask for (counter)examples. You can take any topological spaces you like (but it is also possible to find such examples in standard topology on $\mathbb{R}$). Please justify your examples carefully.

1. Let X be a topological space. Prove that

(a) $\emptyset$ and X are closed;

(b) if $V_1, \ldots, V_k$ is a finite collection of closed sets, then $\cup_{i=1}^k V_k$ is closed;

(c) if $\{V_\alpha\}$ is an arbitrary (possibly infinite) collection of closed sets, then $\cap_\alpha V_\alpha$ is closed.

(d) Give an example of a topological space and an infinite collection of closed sets whose union is *not* closed.

2. Determine whether the sets A, B, and C are open, closed, both open and closed, or neither open nor closed in the given topologies. In each part, find the interior and the closure of the given set. Justify all your answers carefully.

(a) $A = \{a, b\}$ in the topological space $X = \{a, b, c\}$ with topology $\{\emptyset, \{b\}, \{c\}, \{b, c\}, X\}$;

(b) $B = \{(-1)^n/n \mid n = 1, 2, 3 \ldots\}$, in $\mathbb{R}$ with standard topology;

(c) $C = \{(x, y) \in \mathbb{R}^2 : x \neq 0\}$ (the complement of the y-axis in $\mathbb{R}^2$) in $\mathbb{R}^2$ with standard topology;

3. Let X, Y be topological spaces, $f : X \to Y$ a function.

(a) Show that f is continuous *if and only if* $f^{-1}(V)$ is closed (in X) for every closed $V \subset Y$.

(b) Suppose $f : X \to Y$ is continuous. Show that $f^{-1}(IntA) \subset Intf^{-1}(A)$ for every $A \subset Y$. (A is an arbitrary subset, not necessarily open or closed.)

(c) Find an example of a continuous function $f : X \to Y$ and $A \subset Y$ such that $f^{-1}(IntA) \neq Intf^{-1}(A)$.

4. By definition, the closure ClA of a subset A of a topological space is the intersection of all closed sets containing A. Working from this definition, prove that A is closed *if and only if* $ClA = A$.

5. Let A, B be subsets of a topological space X.

(a) Find an example of X, A, B where $Cl(A \cap B) \neq Cl(A) \cap Cl(B)$.

(b) Show that $Cl(A \cap B) \subset Cl(A) \cap Cl(B)$ for all $A, B \subset X$ (in every topological space).

6. Let X be a topological space, $A \subset X$. Let $\mathcal{B}$ be a basis that generates the topology on X. Suppose $x \in X$.

(a) Show that $x \in IntA$ *if and only if* there exists $B \in \mathcal{B}$ such that $x \in B \subset A$.

(b)* Show that $x \notin ClA$ *if and only if* there exists $B \in \mathcal{B}$ such that $x \in B$ and $B \cap A = \emptyset$.

Part (b) is an optional challenge (do it if you can; no points will be given for this part).