MAT 364, Homework 3 due 9/15

Only three questions this week, but Question 1 has several parts and requires a lot of work. (This question will be worth more points.)

1. (a) For each of the following collections of subsets of $\mathbb{R}$, check whether the given collection forms a basis of some topology on $\mathbb{R}$. (In other words, check if the conditions of Definition 1.5 are satisfied.) Justify your answers.

(i) $\mathcal{B}_1 = \{(m, n) \mid m, n \in \mathbb{Z},\ m < n\}$ (all open intervals with integer endpoints);

(ii) $\mathcal{B}_2 = \{[a, b] \mid a, b \in \mathbb{R},\ a < b\}$ (all closed intervals);

(iii) $\mathcal{B}_3 = \{(a, b] \mid a, b \in \mathbb{R},\ a < b\}$ (all half-open intervals, open on the left, closed on the right);

(b) For each of the collections $\mathcal{B}_i$ that does form a basis, consider the topology $\mathcal{T}_i$ generated by this basis, and answer the following questions: is $(0, +\infty)$ open in this topology? Is the interval $(0, \sqrt{2})$ open?

(c) Compare each of the topologies $\mathcal{T}_i$ from part (b) to the standard topology on $\mathbb{R}$: is $\mathcal{T}_i$ the same as the standard topology? are these topologies comparable? is $\mathcal{T}_i$ (strictly) finer or (strictly) coarser than the standard topology?

2. Let $Y = \{a, b, c, d, e\}$ be a set, with topology given by

$$\begin{aligned}\mathcal{T} = \{&\emptyset, \{a\}, \{a, b\}, \{a, b, c\}, \{a, b, c, d\}, \{b\}, \{b, c\}, \{b, c, d\}, \{c\}, \{c, d\}, \{d\},\\ &\{a, c\}, \{a, d\}, \{b, d\}, \{a, c, d\}, \{a, b, d\}, \{a, b, c, d, e\}\}.\end{aligned}$$

(a) Find a basis $\mathcal{B}$ of this topology consisting of 5 non-empty sets. Explain why your collection $\mathcal{B}$ satisfies the conditions of Definition 1.5. Show that the basis $\mathcal{B}$ generates the topology $\mathcal{T}$ given above.

(b) Give an example of a non-constant continuous function $f : \mathbb{R} \to Y$, where $\mathbb{R}$ has its standard topology. Justify continuity using Theorem 4.3. Explain briefly why using this theorem is preferable to working with preimages of all open sets.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = |x|$. Give a topological proof that this function is continuous in the standard topology on $\mathbb{R}$, working with the basis. (Use Theorem 4.3). Explain briefly why using this theorem is preferable to working directly from the definition.

Optional challenge question: question 1.16 from the textbook.

Optional reading: Theorem 1.13. (Please give a direct proof for Question 1.16 without referring to this theorem; the theorem generalizes the statement of the question.)

If you solve this question, it's a good idea to write and submit your solution; it will be checked (but no points will be given).