MAT 364, Homework 2 due 9/8

In this homework, please answer the following questions about continuity, justifying your answers directly from the topological definition of continuous functions. Your arguments should involve preimages of open sets rather than references to any theorems.

1. In this question, consider the set $X = \{a, b, c\}$ with two different topologies, $\mathcal{T}_1 = \{\emptyset, \{a\}, \{a, b\}, \{a, b, c\}\}$, $\mathcal{T}_2 = \{\emptyset, \{a\}, \{a, b, c\}\}$. For (c) and (d), consider also a two-element set $Z = \{0, 1\}$, with discrete topology $\mathcal{T} = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$. (You don't have to check that these collections of subsets form topologies.)

(a) Let $f : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$ be the identity function, $f(x) = x$. Is f continuous?

(b) Now consider the same function $f(x) = x$ but as $f : (X, \mathcal{T}_2) \to (X, \mathcal{T}_1)$. Is f continuous? Remember that continuity is affected by the choice of topology, so this question is different from part (a).

(c) Is there a function $g : X \to Z$ such that $g : (X, \mathcal{T}_1) \to (Z, \mathcal{T})$ is continuous but $g : (X, \mathcal{T}_2) \to (Z, \mathcal{T})$ is not continuous? Give an example (with proof) or show that such g does not exist.

(d) Is there a function $h : X \to Z$ such that $h : (X, \mathcal{T}_2) \to (Z, \mathcal{T})$ is continuous but $h : (X, \mathcal{T}_1) \to (Z, \mathcal{T})$ is not continuous? Give an example (with proof) or show that such h does not exist.

2.(a) Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x + 1 & \text{if } x \geq 0 \\ -x - 1 & \text{if } x < 0. \end{cases}$$

Is this function continuous if $\mathbb{R}$ is equipped with standard topology (both domain and target)? Prove your answer using the topological definition of continuity; justify carefully why the sets in your argument are open or not open.

(b) Now suppose that $\mathbb{R}$ is equipped with the topology of finite complements (both the domain and the target), and consider the same function f. Will this function be continuous?

3. Let $Y = \{a, b, c\}$ be a three-point set, $\mathcal{T} = \{\emptyset, \{a\}, \{a, b\}, \{a, b, c\}\}$.

(a) Give an example of a non-constant continuous function $f : \mathbb{R} \to Y$, where $\mathbb{R}$ has its standard topology. Justify continuity carefully.

(c) Will your function remain continuous if instead you take discrete topology on $\{a, b, c\}$? Why?

4. Suppose X is a space with topologies $\mathcal{T}_1$ and $\mathcal{T}_2$. Consider the identity function $i(x) = x$, and assume that we take the topology $\mathcal{T}_1$ on the domain and the topology $\mathcal{T}_2$ on the target. In other words,

$$i : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$$

if the topologies are incorporated into the notation.

Suppose the function i is continuous with respect to these topologies.

Is the topology $\mathcal{T}_1$ finer or coarser than $\mathcal{T}_2$? Justify your answer carefully.