MAT 364, LECTURE NOTES ON GRAPHS AND SURFACES.

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These notes are meant as a supplement to the textbook. Please still read the textbook!

1. Graphs

A *graph* is a topological space that consists of a finite number of vertices and edges. Vertices are points, edges are homeomorphic to closed intervals, and each edge connects a pair of vertices. Edges are not allowed to intersect in their interior (they can only meet at their endpoints at vertices).

More formally, we use quotient topology to describe topology on a graph: a graph G is a quotient space of the space X of *disjoint* vertices and edges. Think of X, for example, as a subset of $\mathbb{R}^2$ with subspace topology. The quotient is formed by "gluing" the endpoints of intervals to points via the appropriate equivalence relation (when the endpoint of the edge is the corresponding vertex). See example 3.17 in the textbook (Chapter 3), and more in the Chapter 13 (Graphs).

We will not allow loop edges (an edge connecting a vertex to itself), and assume that any pair of vertices is connected by at most 1 edge (no multiple edges connecting the same two vertices). Such graphs are called *simple* in the textbook; the textbook considers non-simple graphs as well.

Definition 1.1. A degree of a vertex is a number of edges meeting at that vertex.

There are two useful equivalence relations on the set of graphs: (1) we can talk about graphs that are homeomorphic (as topological spaces), or (2) graphs that are isomorphic (see below).

Definition 1.2. Two graphs G_1 and G_2 are *isomorphic* if there is a bijection between the vertices of G_1 and vertices of G_2, and another bijection between the edges of G_1 and the edges of G_2, so that if two vertices are connected by an edge in G_1, then the corresponding edge connects the corresponding pair of vertices in G_2.

Theorem 1.3. *(1) If two graphs are isomorphic, then they are homeomorphic.*

(2) If graphs G_1 and G_2 are homeomorphic, then they can be converted into isomorphic graphs G_1' and G_2' by adding extra vertices in the interior of edges (and creating more edges as each old edge gets separated by new vertices into several new edges).

Note that the addition of vertices may be necessary: think of a graphs homeomorphic to a closed interval, formed by a chain of edges. They may consist of different number of edges (longer or shorter chains). The above theorem can be proved by thinking about the quotient topology; we won't give a detailed proof. (Part 1 is easy, part 2 is harder).

We will often consider connected graphs. One can think about connectedness in the topological sense, or just in terms of walking from vertex to vertex along edges:

Theorem 1.4. *The following are equivalent:*

(1) The graph G is connected as a topological space (i.e. there is no separation);

(2) For any pair of vertices $u, w \in G$, there is a sequence of vertices $v_0 = u, v_1, \ldots, v_m = w$, so that there is an edge of G connecting v_{i-1} to v_i, $i = 1, \ldots, m$.

We proved this theorem in class. I'll probably add the proof later.

Definition 1.5. A *cycle* is a sequence of vertices $v_1, v_2, \ldots, v_n$ that are all distinct and such that v_1 is connected by an edge to v_2, v_2 to v_3, and so on, v_{n-1} is connected to v_n, and v_n is connected to v_1. Usually we refer to vertices + their connecting edges as a cycle.

Definition 1.6. A graph is called *acyclic* if it has no cycles.

Definition 1.7. A connected acyclic graph is called a *tree*.

Lemma 1.8. *If every vertex of a graph G has degree at least 2, then G has a cycle.*

Proof. Starting at some vertex, walk along an edge to the second vertex, then from the second vertex to the third, and so on, using a new edge every time. (Note that there must be at least 3 vertices.) Since each vertex has at least 2 edges, if you walk to a vertex along one edge, there is always another edge out of that vertex. However, sooner or later you must come back to one of the vertices that you already visited: the process cannot continue indefinitely with new vertices since the graph is finite. The first time you revisit a vertex, you get a cycle. (The cycle wouldn't necessarily start with vertex 1: for example, if the sequence of vertices is 1, 2, 3, 4, 5, 6, 7, 4, ..., then the cycle involves vertices 4, 5, 6, 7.)

□

Theorem 1.9. *If G is a tree with at least 2 vertices, then G has a vertex of degree 1. (That is, a vertex with only one edge out of it.)*

Proof. Follows from the lemma: since G has no cycles, there must be a vertex of degree 0 or 1; degree 0 isn't possible since G is connected (and has at least 2 vertices by assumption). □

Let V denote the number of vertices, E the number of edges of a graph.

Theorem 1.10. *If G is a tree, then $V = E + 1$.*

Proof. The claim is easy to check for trees with 1, 2, or 3 vertices. Arguing by induction, suppose that the claim is already established for all trees with at most n vertices. Suppose that G is a tree with $(n+1)$ vertices. From the previous theorem, G has a vertex, v, with $\deg v = 1$. Remove v and its edge from G to form a new graph T. (The other endpoint of the removed edge is not removed). Observe that T is a tree: there are no cycles because there weren't any in G, and T is connected: any two vertices $u, w \in T$ can be connected by a path of edges on G, and even if this path detours into the dead-end vertex v in G, this detour can be avoided to make a path from u to w in T. The identity $V = E + 1$ then holds for T by the induction hypothesis, and therefore it holds for G, since G differs from T by one edge and one vertex. □

Definition 1.11. The quantity $V - E$ is called the *Euler characteristic* of the graph. The usual notation is the Greek letter χ.

The theorem above says that the Euler characteristic of a tree always equals to 1. Here are a few more statements about the Euler characteristic of graphs.

Theorem 1.12. *If G is a connected graph, then $V \leq E + 1$; in other words, $\chi(G) \leq 1$.*

Proof. If G is tree, $V = E + 1$ as already established. Suppose that G is not a tree, and therefore G has cycles. Remove an edge (but not any vertices) from any cycle in G to form a new graph, G'. Then G' is connected: since G is connected, any pair of vertices u, w can be connected by a path of edges in G. If this path uses the removed edge, we can still reach w from u in G' via a detour that uses the remaining edges of the cycle. (Make a picture to see this.) If G' is a tree, then $\chi(G') = 1$, and $\chi(G) < \chi(G')$ since G has the same vertices as G' and one more edge. If G' is not a tree, then we continue the process of removing edges from cycles, keeping the graph connected. Since the graph only has finitely many edges, after finitely many steps we will get a graph T which is a tree. Because G has the same vertices as T but more edges, we have $\chi(G) < \chi(T) = 1$, as desired. □

The proof yields several useful facts:

Corollary 1.13. *If a graph G is connected but has cycles, then $\chi(G) = V - E < 1$.*

Corollary 1.14. *A connected graph with $V = E + 1$ (equivalently, $\chi = 1$) must be a tree.*

Corollary 1.15. *Every connected graph G has a* maximal tree*: a tree that consists of all the vertices and some of the edges of G. (A maximal tree is not unique, there might be several different ways to obtain a tree from G).*

It becomes clear from the previous theorem that the "extra" edges are responsible for the cycles in a connected graph. Setting a count of cycles, however, is a bit problematic, since some cycles can be created as a combination of other cycles (going through several cycles in order), and it's hard to tell which cycles should be counted as "essential" and which ones arise as a combination. (How many cycles do the edges of a tetrahedron form? Only 3 edges need to be discarded to make a tree.) The count of cycles becomes more straightforward if the graph is *planar*.

Definition 1.16. A graph G is called *planar* if it can be drawn in $\mathbb{R}^2$ without creating any additional intersections of edges. (In other words, $G \subset \mathbb{R}^2$, and its topology as a graph is the same as subpace topology.)

We will later see that some graphs are *not* planar.

Instead of cycles, we will now look at *faces* for planar graphs. We assume that the graph is a subset of the plane.

Definition 1.17. For a planar graph $G \subset \mathbb{R}^2$, a *face* is a region in $\mathbb{R}^2$ bounded by edges of the graph, so that no other edges are contained inside this region. We will also include the unbounded region of $\mathbb{R}^2$ as a face. (See pics on p.424-425 of the textbook.)

Let F denote the number of faces; V stands for vertices, E for edges as before.

Theorem 1.18. (Euler's formula) *For every connected planar graph,*

$$V - E + F = 2.$$

Proof. If the graph is a tree, there is only one face (the unbounded region): a bounded face would produce a cycle by following the sequence of edges on its boundary. Since $V - E = 1$ for trees, we get that $V - E + F = 2$ for the case of trees.

If the graph G is not a tree, there will be some bounded faces: a cycle bounds a bounded region (which can be a single face, or consist of several faces if there are edges going though this region). Find an edge that separates two bounded faces, or an edge that separates the unbounded face from a bounded one. Remove this edge; notice that two faces merge (and become one face) as a result, so both the number of edges and the number of faces decrease by 1. Continue this process (finitely many times) until no bounded faces remain, and the resulting graph T is a tree. Then, $V - E + F = 2$ for T. The graph G has k more edges and k more faces than T; it follows that the formula holds for G as well. □

In a similar way, we can place planar graphs on the sphere. A special case of this is when the sphere is separated by edges and vertices into a number of polygonal faces, each face bounded by edges, so that two faces can touch at a vertex or share an edge (or be disjoint). Think about a soccer ball for an example. (For another example, blow air into a cube to make its surface spherical; the sphere would then be separated into squares corresponding to faces of the cube.)

Theorem 1.19. (Euler's formula for the sphere) *If a sphere is cut into polygonal faces by edges and vertices as above, then*

$$V - E + F = 2.$$

Proof. This is the same fact as Euler's formula for planar graphs: if you delete one of the faces on the sphere, then the rest of it (which is homeomorphic to a disk) can be stretched and placed on the plane, so that the edges and vertices form a planar graph, and the deleted face in the sphere corresponds to the unbounded face in $\mathbb{R}^2$. □

We can now define the Euler characteristic of any surface cut up into vertices, edges and faces as above (or any space formed by polygonal faces, edges and vertices in a similar way):

$$\chi = V - E + F.$$

Even more generally, the Euler characteristic can be defined for spaces in higher dimensions, built in a similar way from polyhedral building blocks homeomorphic to closed disks in $\mathbb{R}^k$: χ is the alternating sum of the number of blocks in each dimension.

Theorem 1.20. *The Euler characteristic is a topological invariant: it depends only on the topological space, not on the way the space is cut into faces, edges, and vertices. If two spaces X and Y are homeomorphic, then $\chi(X) = \chi(Y)$.*

This is a difficult fact which we won't prove in this course. Note, however, that we did prove invariance for the sphere, and computed $\chi(S^2) = 2$.

Example 1.21. The Euler characteristic of the torus is $\chi(T) = 0$. Compute this by drawing a picture.