MAT 364 - Homework 9 Solutions

Exercise 3.15– Show that if X is a topological space consisting of a finite number of points, with any topology, then X is compact.

Proof: As X is finite, we can write $X = \{x_1, x_2, \ldots, x_n\}$. Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ be any open cover of X. Note that we do not assume anything about this cover - the index set I is an arbitrary set, and we make no assumption that the U_α are all unique sets. We only know that each U_α is an open set and that the union $\cup_{\alpha \in I} U_\alpha = X$.

Then for each $x_i \in X$, there exists an open set U_{α_i} from the cover such that $x_i \in U_{\alpha_i}$. Consider the finite subcollection of $\mathcal{U}$ given by $\{U_{\alpha_i}\}_{i=1}^n$. Clearly the union $\cup_{i=1}^n U_{\alpha_i} \subset X$. Given $x_i \in X$, we have $x_i \in U_{\alpha_i} \subset \cup_{i=1}^n U_{\alpha_i}$, hence $X \subset \cup_{i=1}^n U_{\alpha_i}$, and therefore $X = \cup_{i=1}^n U_{\alpha_i}$.

Thus the collection $\{U_{\alpha_i}\}_{i=1}^n$ is a finite subcover of $\mathcal{U}$. Then every open cover of X has a finite subcover, hence X is compact.

Exercise 3.16 – Show that any space X with the indiscrete topology is compact.

Proof: Let X be a space with the indiscrete topology, that is, the only open sets in X are X and $\emptyset$. Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ be an open cover of X. At least one of these U_α is nonempty, for if they were all empty then $\cup U_\alpha = \emptyset$, and we would have that $\mathcal{U}$ is not a cover of X. So there exists some fixed $\beta \in I$ such that U_β is nonempty. Since U_β is a nonempty open set and X has the indiscrete topology, we have that $U_\beta = X$.

Then the collection $\{U_\beta\}$, consisting of a single element, is a finite subcover of $\mathcal{U}$. Clearly it is finite, consists of open sets, and the union of all elements in this subcover is X, which covers X. Thus any open cover has a finite subcover, hence X is compact.

Exercise 3.22 – Show that $\mathbb{N}$ is connected in the finite complement topology.

Proof: Recall that the finite complement topology on $\mathbb{N}$ is that a non-empty subset $U \subset \mathbb{N}$ is open if and only if the complement $\mathbb{N} - U$ is a finite set.

Assume by contradiction that $\mathbb{N}$ is not connected - that is, that there exists A, B nonempty, open sets such that $A \cup B = \mathbb{N}$ and $A \cap B = \emptyset$. Note that this implies that $\mathbb{N} - A = B$ and $\mathbb{N} - B = A$. Because A is open and we are using the finite complement topology, the complement $\mathbb{N} - A = B$ is finite. Similarly, because B is open we have that $\mathbb{N} - B = A$ is finite. Thus A, B are both finite sets, so their union $A \cup B = \mathbb{N}$ is a finite set. But $\mathbb{N}$ is infinite, and we have a contradiction.

Exercise 3.25 – Prove that if X is a Hausdorff space, Y is a compact subset of X, and $x \in X - Y$, then there are disjoint open sets U and V in X such that $x \in U$ and $Y \subset V$.

Proof: Consider any $y \in Y$. Then because X is Hausdorff, there exist disjoint open sets U_y, V_y such that $x \in U_y$ and $y \in V_y$. Letting y vary, this gives two collections of open sets, $\{U_y\}_{y \in Y}$ and $\{V_y\}_{y \in Y}$.

Observe that this second collection is an open cover of Y - it is clearly a collection of open sets, and it contains every $y \in Y$ by construction. Since Y is compact, this open cover has a finite subcover. That is, there exist $\{y_1, \ldots, y_m\}$ a finite collection of points in Y such that $\cup_{i=1}^m V_{y_i} \supset Y$.

Let $V = \cup_{i=1}^m V_{y_i}$. This is a union of open sets in X, hence is open, and from above we have that $Y \subset V$. Let $U = \cap_{i=1}^m U_{y_i}$, so that U is the intersection of the U_{y_i} that are associated to the V_{y_i} in the finite subcover. Then U is a finite intersection of open sets, hence open, and also $x \in U$ since $x \in U_y$ for all $y \in Y$.

It only remains to check that $U \cap V = \emptyset$. Assume by contradiction that there exists some point $z \in U \cap Y$. Since $Y = \cup_{i=1}^m U_{y_i}$, $z \in Y$ implies that $z \in U_{y_i}$ for some fixed i. Similarly, $z \in U$ implies that $z \in U_{y_i}$. Then $z \in U_{y_i} \cap V_{y_i}$, but these open sets are disjoint. With this contradiction, we must have that $U \cap V = \emptyset$.

Exercise 3.26 – Prove that if X is a Hausdorff space and Y is a compact subset of X, then Y is closed.

Proof: Let $x \in X - Y$. From the previous problem, there exists disjoint open sets U and V such that $x \in U$ and $Y \subset V$. Thus $U \cap Y = \emptyset$, so that $U \subset X - Y$.

Thus for every $x \in X - Y$, there exists an open set U_x containing x with $U_x \subset X - Y$. Let $O = \cup_{x \in X-Y} U_x$. This is a union of open sets, hence is open. Because each $U_x \subset X - Y$, we have that $O \subset X - Y$. On the other hand, for every $x \in X - Y$ we have $x \in U_x \subset O$, so $X - Y \subset O$. Thus $O = X - Y$. Then the complement of Y is an open set, so by definition Y is closed.

Question 1 – Suppose $f : X \to Y$ is a continuous function which is onto.
Part A: If X is Hausdorff, must Y always be Hausdorff?

Solution to Part A: Y need not be Hausdorff. As a counterexample, consider any set X with more than 1 point. Let X_1 be that set equipped with the discrete topology, which is Hausdorff as shown in class. Let X_2 be that set equipped with the indiscrete topology, which is not Hausdorff as shown in class. Let $f : X_1 \to X_2$ be the identity function. Then this is a surjection (the identity function is always a bijection) and is also continuous, as any function with discrete domain is continuous (or any function with indiscrete range is continuous).

Part B: If Y is Hausdorff, must X always be Hausdorff?

Solution to Part B: X need not be Hausdorff. As a counterexample, let $X = \{a, b, c\}$ be given the topology $\mathcal{T}_X = \{\emptyset, \{a\}, \{b, c\}, X\}$. Note that this is a valid topology, and this topology is not Hausdorff, as any open set that contains b also contains c. Let $Y = \{d, e\}$, with the discrete topology, which is Hausdorff. Let $f : X \to Y$ be defined by $f(a) = d$ and $f(b) = f(c) = e$. This function is surjective, and it is not hard to check the preimages of all open sets to see that this function is also continuous.

Question 2 – Let $A = \{1, 1/2, 1/3, \ldots, 1/n, \ldots\}$ and let $B = A \cup \{0\}$. Is A compact? Is B compact? Argue directly from the definitions.

Proof: A is not compact. To prove this, we need to find an open cover that does not have any finite subcover. For any $n \geq 1$, observe that $\left(\frac{1}{n+1/2}, \frac{1}{n-1/2}\right) \cap A = \frac{1}{n}$, so that in fact every singleton set $\{1/n\}$ is open in the subspace topology on A. The collection of all of these singleton sets is therefore an open cover of A, and cannot have a finite subcover. The union of a finite number of points from this cover is only a finite set, and a finite set cannot contain an infinite set. Another open cover without a finite subcover is given by the collection $\{(1/n, 2)\}_{n=2}^{\infty}$ - observe that because these open sets are nested, any finite collection of them is contained in $(1/m, 2)$ for some large fixed m, which will not cover all of A.

On the other hand, B is compact. Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ be any open cover of B. Because this collection is a cover, there exists some U_α from this cover such that $0 \in U_\alpha$. Note that U_α is an open set in $\mathbb{R}$, so by definition of the topology on $\mathbb{R}$ is contains some open interval $(-\epsilon, \epsilon)$ around 0.

Observe that there exists some $N > 1$ such that $1/N \geq \epsilon$ but $\frac{1}{N+1} < \epsilon$. Thus, the set $(-\epsilon, \epsilon)$ will contain all but finitely many points of B - it will only not contain the points $1, 1/2, 1/3, \ldots, 1/N$. Since U_α contains all of $(-\epsilon, \epsilon)$, we have that U_α will contain all contain all but finitely many points of B as well - the only points it might not contain are $1, 1/2, 1/3, \ldots, 1/N$.

But for each of these points, there exists U_{β_i} for $i = 1, \ldots, N$ such that $1/i \in U_{\beta_i}$, since $\mathcal{U}$ must cover all the points of B. Thus, the collection $\{U_\alpha, U_{\beta_1}, \ldots, U_{\beta_N}\}$ covers all the points of B. It is clearly finite, hence is a finite subcover of $\mathcal{U}$. Since every open cover of B has a finite subcover, B is compact.