MAT 364 - Homework 7 Solutions to Selected Exercises

Exercise 2.31, part (2): Give an example of sets $A, B \in \mathbb{R}^2$ such that A, B are connected, but $A - B$ is connected.

Let $A = D^2((0,0), 1)$ and $B = \{(x, 0) \in \mathbb{R}^2 : -1 < x < 1\}$. The set A is convex, hence is connected by Question 4 on this homework. (Compare with Q5, part (ii) on this homework). The set B is homeomorphic to the open interval $(-1, 1)$ with the standard topology as a subspace of $\mathbb{R}$, and it has been shown in class that open intervals of $\mathbb{R}$ are connected in the standard topology.

The set $A - B$ can be written as $A - B = U \cup V$, where $U = A \cap \{(0, y) \in \mathbb{R}^2 : y > 0\}$ (the "upper half plane") and $V = A \cap \{(0, y) \in \mathbb{R}^2 : y < 0\}$ (the "lower half plane"). Note that A and both "half planes" are open in $\mathbb{R}^2$, so the intersections U, V are open in $\mathbb{R}^2$, hence are open relative to $A - B$. Both A, B are nonempty - they contain $(0, 1/2)$ and $(0, -1/2)$, respectively. Finally, they are disjoint, as no point (x, y) can have both $y > 0$ and $y < 0$. Thus $A - B$ can be written as the disjoint union of two nonempty open sets, hence $A - B$ is not connected.

Note: When you are asked to give an example in a question, you must explain why your answer is an example to get full credit. Thus for each part of this question, you had to show three things - that A, B are connected (in parts 1,2) or disconnected (part 3) and also that the sets $A \cap B$, $A - B$, and $A \cup B$ are connected or not, as appropriate. Simply stating your choice of A and B is not sufficient.

Question 1: Let $X = \{a, b, c, d\}$. Give examples of (1) two different topologies on X such that both spaces are connected but not homeomorphic (2) two different topologies on X such that both spaces are not connected but homeomorphic.

Solution: A solution will only be given for part 1, as the second part is very similar. For the first part, you could choose the topologies $\mathcal{T}_1 = \{X, \emptyset\}$ and $\mathcal{T}_2 = \{X, \emptyset, \{a\}\}$. To receive full credit, you had to explain why your choices were topologies - note that $\mathcal{T}_1$ is the indiscrete topology, which we know is a valid topology, and it is not hard to check that $\mathcal{T}_2$ is also a valid topology by considering unions and intersections.

Next, we show that $(X, \mathcal{T}_1)$ and $(X, \mathcal{T}_2)$ are connected. Recall that we've already shown on a previous homework that any space with the indiscrete topology is connected. To see that $(X, \mathcal{T}_2)$ is connected, we can observe that the only possible pairs of disjoint open sets are $\emptyset, \{a\}$ and $\emptyset, X$. In each case one of the pairs in the set is empty, so we cannot partition X into two disjoint, nonempty open sets. Therefore no disconnection on X is possible with the $\mathcal{T}_2$ topology, hence the space is connected.

Finally, we show that these spaces are not homeomorphic. Assume by contradiction that there is a homeomorphism $f : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$. Consider $f^{-1}(\{a\})$. Because f is a homeomorphism, we know that it is a bijection, hence the set $f^{-1}(\{a\})$ contains exactly one element. We also know that homeomorphisms are continuous, so we have that $f^{-1}(\{a\})$ is open in the $\mathcal{T}_1$ topology. This is a contradiction - there are no one-element open sets in the $\mathcal{T}_1$ topology. Thus our original assumption that a homeomorphism exists is incorrect.

Note 1: We could prove that $(X, \mathcal{T}_1)$ and $(X, \mathcal{T}_2)$ are not homeomorphic another way. The main fact is that if $(X, \mathcal{T}_1)$ and $(X, \mathcal{T}_2)$ are homeomorphic, then their topologies have to have the same cardinality. (Try to prove this - assume that a homeomorphism f exists, and use the fact that it is bijective with f, f^{-1} continuous to show that it induces a bijection between the sets $\mathcal{T}_1$ and $\mathcal{T}_2$). In our example, we have that $|\mathcal{T}_1| = 2$ and $|\mathcal{T}_2| = 3$, so the topologies are not homeomorphic.

Note 2: We showed that *no* homeomorphism could exist, not just that a particular function is not a homeomorphism. For example, one could show that the identity function is not a homeomorphism, perhaps by showing that it is not continuous. After showing this, though, it could still be possible that some other homeomorphism could exist - perhaps the function that takes $a \mapsto b, b \mapsto a$, and fixes c, d could be a homeomorphism. Instead of trying to analyze all of the possible functions on X and check one-by-one that they are not homeomorphisms, we argue as above.

Question 3: Let $\{U_\alpha\}_{\alpha \in A}$ be a collection of connected subsets of a topological space X. Assume that $\bigcap_{\alpha \in A} U_\alpha$ is not empty. Show that the union $\bigcup_{\alpha \in A} U_\alpha$ is connected.

Solution: Assume by contradiction that $U = \bigcup_{\alpha \in A} U_\alpha$ is not connected. That is, there exists $A, B \subset U$ such that $A \cup B = U$, both sets are nonempty, A, B are disjoint, and A, B are relatively open as subsets of U.

Because the intersection $\bigcap_{\alpha \in A} U_\alpha$ is nonempty, there exists some point $z \in \bigcap_{\alpha \in A} U_\alpha$. Without loss of generality, we assume that $z \in A$. Because B is nonempty, we can also pick some $w \in B$. Now $w \in U$, so there exists some U_α such that $w \in U_\alpha$ since U is a union of the U_α. We also have that $z \in U_\alpha$.

Then let $\bar{A} = A \cap U_\alpha$ and $\bar{B} = B \cap U_\alpha$. Note that these sets are both nonempty, as $z \in \bar{A}$ and $w \in \bar{B}$. Both $\bar{A}$ and $\bar{B}$ are relatively open in U_α, since they are the intersection of the open sets A, B with U_α. Their union is

$$\bar{A} \cup \bar{B} = (A \cap U_\alpha) \cup (B \cap U_\alpha) = (A \cup B) \cap U_\alpha = U \cap U_\alpha = U_\alpha$$

and their intersection is

$$\bar{A} \cap \bar{B} = (A \cap U_\alpha) \cap (B \cap U_\alpha) = (A \cap B) \cap U_\alpha = \emptyset \cap U_\alpha = \emptyset$$

that is, $\bar{A}$ and $\bar{B}$ are disjoint.

Thus we can write U_α as the union $U_\alpha = \bar{A} \cup \bar{B}$ of disjoint, nonempty open sets, so that U_α is not connected. But this contradicts the hypothesis that all the U_α are connected. Therefore our original assumption is incorrect, that is, the union $\cup_{\alpha \in A} U_\alpha$ is connected.

Question 4: Let $W \subset \mathbb{R}^n$ be a convex subset - that is, if $x, y \in W$, then the line segment joining x to y in $\mathbb{R}^n$ is also contained in W. Show that W is connected.

Solution: Assume by contradiction that W is not connected. That is, there exists $A, B \subset U$ such that $A \cup B = W$, both sets are nonempty, A, B, are disjoint, and A, B are relatively open as subsets of W.

Because A and B are nonempty, we can choose points $a \in A$ and $b \in B$, and let L be the line segment joining those two points. Let $\bar{A} = A \cap L$ and $\bar{B} = B \cap L$. Using similar arguments as in the previous problem, we can show that the sets $\bar{A}, \bar{B}$ are disjoint, nonempty subsets of W that are relatively open in W. We also have that

$$\bar{A} \cup \bar{B} = (A \cap L) \cup (B \cap L) = (A \cup B) \cap L = W \cap L = L.$$

The last equality in the line above is where we use the fact that W is convex. Convexity gives that $L \subset W$, and therefore $W \cap L = L$.

Thus we can write L as the union $L = \bar{A} \cup \bar{B}$ of disjoint, nonempty open sets, so that L is not connected. But L is a line segment in $\mathbb{R}^n$, hence homeomorphic to a closed interval in $\mathbb{R}$, and we know that closed intervals are connected. Thus L is connected, and we have a contradiction. Our original assumption was wrong, and so W is connected.

Question 5: Determine with proof the connectedness of the the plane, the open disk, the plane with a line or point removed, and the rationals in $\mathbb{R}$.

Solution: For (i), (ii), which are $\mathbb{R}^2$ and the open disk $D^2(x, r)$ in $\mathbb{R}^2$, both with the standard topologies, we can observe that both spaces are convex, hence connected by the previous question.

For (iii), we consider $\mathbb{R}^2$ with the x-axis removed. We can write this set as $A \cup B$, where $A = \{(x, y) \in \mathbb{R}^2 : y > 0\}$ and $B = \{(x, y) : y < 0\}$, the upper and lower half planes. These sets are clearly nonempty, disjoint, and open in $\mathbb{R}^2$, hence open in $\mathbb{R}^2 - \{x - axis\}$, so we have that $\mathbb{R}^2 - \{x - axis\}$ is not connected.

For (iv), consider A, B the upper and lower half planes as defined above, and let C, D be the "left" and "right" half planes, $C = \{(x, y) : x < 0\}$, $D = \{(x, y) : x > 0\}$. Each of A, B, C, D is connected, as they are convex. Note that $A \cap C$ is nontrivial (it is the second quadrant in the plane) so therefore $A \cup C$ is connected. Then $(A \cup C) \cap B$ is nontrivial (the third quadrant in the plane) so $(A \cup B) \cup C = A \cup B \cup C$ is connected. A similar argument with D shows that $A \cup B \cup C \cup D$ is connected, but this set is just $\mathbb{R}^2 - \{(0, 0)\}$, so the plane with the origin removed is connected.

For (v), let $A = \mathbb{Q} \cap (-\infty, \alpha)$ and $B = \mathbb{Q} \cap (\alpha, \infty)$ for α an *irrational* number. Note that A and B are easily seen to be disjoint, open in the relative topology on $\mathbb{Q}$, and nonempty. We also have $A \cup B = \mathbb{Q}$, as $A \cup B = (\mathbb{R}^2 - \{\alpha\}) \cap \mathbb{Q} = \mathbb{Q}$ since α is irrational. Thus we can write $\mathbb{Q}$ as the nonempty disjoint union of two relatively open sets, so $\mathbb{Q}$ is not connected.