MAT 364 - Homework 4 Solutions

<u>Exercise 3.3:</u> Let $X = \{x, y, z\}$, the set with 3 elements. Figure out as many topologies for X as you can, and represent them both schematically and by listing their open sets.

Solution: For this solution, I'll only list out the open sets of each topology, as generating the necessary figures is hard to do on a computer.

Since X has three elements, it has $2^3 = 8$ possible subsets, namely

$$P(X) = \{\emptyset, \{x\}, \{y\}, \{z\}, \{x, y\}, \{x, z\}, \{y, z\}, X\}.$$

(The set of all subsets of X is called the "power set" of X.) A topology on X is a subset of $P(X)$, subject to certain conditions. *A priori* there are $2^8 = 256$ different subsets of $P(X)$, and each of these is a candidate for a topology on X. We need to find how many of these subsets of $P(X)$ satisfy the conditions of a topology - namely that the subset contains $X, \emptyset$, that it is closed under arbitrary union, and it is closed under finite intersection.

The first condition gives that $X, \emptyset$ must be contained in the topology, and therefore reduces the number of subsets of $P(X)$ we need to test to $2^6 = 64$. One could list out all of these subsets by hand and check each one to see if it is a valid topology, but this is still a lot of work, so it will be easier to work the other way, by starting with the smallest topology possible and adding subsets of X to it one by one, making sure that after each addition we still have a topology.

The topology with the smallest number of elements is the indiscrete topology, $\mathcal{T}_1 = \{X, \emptyset\}$.

From there, we see what happens when we add sets to this topology. We observe that if we add any other subset of X to $\mathcal{T}_1$, we still get a valid topology, because the new set will still be closed under union and intersection. This gives the topologies.

$$\begin{array}{rclrcl} \mathcal{T}_2 &=& \{X, \emptyset, \{x\}\} & \mathcal{T}_5 &=& \{X, \emptyset, \{x, y\}\} \\ \mathcal{T}_3 &=& \{X, \emptyset, \{y\}\} & \mathcal{T}_6 &=& \{X, \emptyset, \{x, z\}\} \\ \mathcal{T}_4 &=& \{X, \emptyset, \{z\}\} & \mathcal{T}_7 &=& \{X, \emptyset, \{y, z\}\} \end{array}$$

We now consider adding more sets to the topology $\mathcal{T}_2$. We first consider what happens if we add only 2-element sets to this topology. Adding any of $\{x, y\}, \{x, z\}$, or $\{y, z\}$ still gives a valid topology, so we have found the topologies.

$$\mathcal{T}_8 = \{X, \emptyset, \{x\}, \{x, y\}\} \qquad \mathcal{T}_9 = \{X, \emptyset, \{x\}, \{x, z\}\} \qquad \mathcal{T}_{10} = \{X, \emptyset, \{x\}, \{y, z\}\}$$

We can continue adding sets to these topologies. For example, we could add the set $\{x, z\}$ to $\mathcal{T}_8$, yielding the topology $\mathcal{T}_{11} = \{X, \emptyset, \{x\}, \{x, y\}, \{x, z\}\}$. If we add the set $\{y, z\}$ to $\mathcal{T}_8$, we will also have to add the set $\{y\}$ to the topology, since $\{y, z\} \cap \{x, y\} = \{y\}$ and topologies must be closed under finite intersection. This will give a topology with two 1-element subsets, which we will discuss more below.

We can apply the same arguments as above to $\mathcal{T}_3$ and $\mathcal{T}_4$, again adding only 2-element sets, or to $\mathcal{T}_5, \mathcal{T}_6$, and $\mathcal{T}_7$, this time adding only 1-element sets. This gives the following list, which includes all topologies that include exactly one single-element set-

$$\begin{array}{rclrclrcl} \mathcal{T}_2 &=& \{X, \emptyset, \{x\}\} & \mathcal{T}_3 &=& \{X, \emptyset, \{y\}\} & \mathcal{T}_4 &=& \{X, \emptyset, \{z\}\} \\ \mathcal{T}_8 &=& \{X, \emptyset, \{x\}, \{x, y\}\} & \mathcal{T}_{12} &=& \{X, \emptyset, \{y\}, \{x, y\}\} & \mathcal{T}_{16} &=& \{X, \emptyset, \{z\}, \{x, y\}\} \\ \mathcal{T}_9 &=& \{X, \emptyset, \{x\}, \{x, z\}\} & \mathcal{T}_{13} &=& \{X, \emptyset, \{y\}, \{x, z\}\} & \mathcal{T}_{17} &=& \{X, \emptyset, \{z\}, \{x, z\}\} \\ \mathcal{T}_{10} &=& \{X, \emptyset, \{x\}, \{y, z\}\} & \mathcal{T}_{14} &=& \{X, \emptyset, \{y\}, \{y, z\}\} & \mathcal{T}_{18} &=& \{X, \emptyset, \{z\}, \{y, z\}\} \\ \mathcal{T}_{11} &=& \{X, \emptyset, \{x\}, \{x, y\}, \{x, z\}\} & \mathcal{T}_{15} &=& \{X, \emptyset, \{y\}, \{x, y\}, \{y, z\}\} & \mathcal{T}_{19} &=& \{X, \emptyset, \{z\}, \{x, z\}, \{y, z\}\} \end{array}$$

We can also add 2-element sets to $\mathcal{T}_5, \mathcal{T}_6$, and $\mathcal{T}_7$. For example, if we add the set $\{x, z\}$ to $\mathcal{T}_5$, then for the new set to still be a topology we need to also add $\{x\}$. This yields the topology $\mathcal{T}_{11}$ that we have already found. Similar arguments show that adding any two-element set to one of $\mathcal{T}_5, \mathcal{T}_6$, or $\mathcal{T}_7$ will require also adding a 1-element subset, and thus will produce one of the topologies listed above. Therefore, an exhaustive list of the topologies on X that contain no 1-element sets is $\mathcal{T}_1, \mathcal{T}_5, \mathcal{T}_6$, and $\mathcal{T}_8$.

We now move on to seeing what happens if we add a 1-element subset to any of the topologies listed above, to try to find a list of all possible topologies that contain exactly two 1-element subsets. For example, we could add the set $\{y\}$ to $\mathcal{T}_2$. This would require also adding the set $\{x, y\}$, so that the topology remains closed under unions. This yields the topology $\mathcal{T}_{20} = \{X, \emptyset, \{x\}, \{y\}, \{x, y\}\}$. We could add 2-element sets to this topology. Adding either $\{x, z\}$ or $\{y, z\}$ yields the topologies $\mathcal{T}_{21} = \{X, \emptyset, \{x\}, \{y\}, \{x, y\}, \{x, z\}\}$ and $\mathcal{T}_{22} = \{X, \emptyset, \{x\}, \{y\}, \{x, y\}, \{y, z\}\}$. If we add another 2-element set to either of these topologies, we would also have to add the set $\{z\}$ so that the topology is closed under intersection, and for now we are only trying to list topologies with exactly two 1-element subsets. Thus we stop here. We can apply the same thinking to adding one-element subsets to any of the topologies listed above, to obtain the following list of all topologies on X that contain exactly two 1-element subsets of X:

$$\begin{aligned}
\mathcal{T}_{20} &= \{X, \emptyset, \{x\}, \{y\}, \{x, y\}\} \\
\mathcal{T}_{21} &= \{X, \emptyset, \{x\}, \{y\}, \{x, y\}, \{x, z\}\} \\
\mathcal{T}_{22} &= \{X, \emptyset, \{x\}, \{y\}, \{x, y\}, \{y, z\}\} \\
\mathcal{T}_{23} &= \{X, \emptyset, \{x\}, \{z\}, \{x, z\}\} \\
\mathcal{T}_{24} &= \{X, \emptyset, \{x\}, \{z\}, \{x, z\}, \{x, y\}\} \\
\mathcal{T}_{25} &= \{X, \emptyset, \{x\}, \{z\}, \{x, z\}, \{y, z\}\} \\
\mathcal{T}_{26} &= \{X, \emptyset, \{y\}, \{z\}, \{y, z\}\} \\
\mathcal{T}_{27} &= \{X, \emptyset, \{y\}, \{z\}, \{y, z\}, \{x, y\}\} \\
\mathcal{T}_{28} &= \{X, \emptyset, \{y\}, \{z\}, \{y, z\}, \{x, z\}\}
\end{aligned}$$

Finally, we consider the topologies that contain all three 1-element subsets. But then because a topologies must be closed under unions, we see that such a topology must contain *all* subsets of X. Thus the final topology is the discrete topology,

$$\mathcal{T}_{29} = \{X, \emptyset, \{x\}, \{y\}, \{z\}, \{x, y\}, \{x, z\}, \{y, z\}\}$$

There are therefore 29 different possible topologies on $X = \{x, y, z\}$.

Remark: One could ask the more general question: given X a set with n elements, how many different topologies are there on X? Although this can be computed by hand for small values of n, there isn't a known formula that gives number topologies on a set as a function of n.

Exercise 3.4: Prove Theorem 2.8 from the text. That is, prove that if X is a topological space, then

(a) $X, \emptyset$ are closed.

(b) The intersection of any collection of closed sets is closed.

(c) The union of any finite collection of closed sets in closed.

Solution: Recall the definition of closed sets for abstract topological spaces - $C \subset X$ is closed if and only if the complement $X - C$ is an open set.

(a) Following the definition, we consider the complements of X and $\emptyset$. We observe that $X - X = \emptyset$. Every topology includes $\emptyset$ as an open set, so X is closed. Similarly, $X - \emptyset = X$, and every topology on X includes X as an open set, so $\emptyset$ is closed.

(b) Let $\{C_\alpha\}_{\alpha \in I}$ be a collection of closed sets for I any index set - in particular, we are not assuming that I is finite or even countable. Let $C = \cap_{\alpha \in I} C_\alpha$. Then the complement of C is

$$X - C = X - \bigcap_{\alpha \in I} C_\alpha = \bigcup_{\alpha \in I} (X - C_\alpha),$$

using DeMorgan's laws. Observe that because each C_α is a closed set, the complement $X - C_\alpha$ is an open set. The definition of a topology says that the union of any collection of open sets is again open, so we have that $\cup_{\alpha \in I}(X - C_\alpha)$ is open. Then $X - C$ is open, so C is a closed set.

(c) Let $C_i, 1 \leq i \leq n$ be a finite collection of closed sets. Let $C = \cup_{i=1}^n C_i$. Then the complement of C is

$$X - C = X - \bigcup_{i=1}^{n} C_i = \bigcap_{i=1}^{n} (X - C_i),$$

again using DeMorgan's laws. Each $X - C_i$ is an open set, since C_i is closed. The definition of a topology says that the intersection of a finite collection of open sets is again open, so we have that $\cap_{i=1}^n (X - C_i)$ is open, hence $X - C$ is open, and therefore C is closed.

Exercise 3.9: Let X, Y be topological spaces, and let $f : X \to Y$ be a function. Show that f is continuous if and only if for every $C \subset Y$ closed, the set $f^{-1}(C) \subset X$ is closed.

Solution: ($\Rightarrow$) Assume that f is continuous. Let $C \subset Y$ be a closed set. Then by definition $Y - C$ is an open set. Then the set $f^{-1}(Y - C)$ is open, because the function f is continuous. We observe that $f^{-1}(Y - C) = X - f^{-1}(C)$, so we have that the complement of the set $f^{-1}(C)$ is open, that is, $f^{-1}(C)$ is closed.

($\Leftarrow$) Assume that for every closed set $C \subset Y$, we have that $f^{-1}(C)$ is closed. Let $U \subset Y$ be an open set. Then by definition the complement $Y - U$ is a closed set, and therefore the preimage $f^{-1}(Y - U)$ is closed. Observing that $f^{-1}(Y - U) = X - f^{-1}(U)$, we have that the complement of $f^{-1}(U)$ is a closed set, that is $f^{-1}(U)$ is open. Thus for every $U \subset Y$ open, we have $f^{-1}(U)$ is open, that is, f is continuous.

Exercise 3.10: Let X be a topological space, and let $A, B \subset X$ be closed subsets of X such that $X = A \cup B$. Let $f : A \to Y$ and $g : B \to Y$ be continuous functions such that for every $x \in A \cap B$, $f(x) = g(x)$. Define $(f \cup g) : X \to Y$ by

$$(f \cup g)(x) = \begin{cases} f(x) & x \in A \\ g(x) & x \in B \end{cases}$$

(a) Prove that $(f \cup g)$ is continuous.

(b) Give an example showing that the hypothesis that both A and B are closed is necessary.

Solution: (a) We will consider the characterization of continuous functions using closed sets from the previous exercise.

Let $C \subset Y$ be a closed set. Then $f^{-1}(C) \subset A$ is closed because f is continuous. The topology on A is the subspace topology, so saying that $f^{-1}(C)$ is closed in A means that there exists $D \subset X$

a closed set such that $f^{-1}(C) = D \cap A$. Now D and A are both closed in X (this is where we use the necessary hypothesis that A is closed), so their intersection is also a closed set in X from exercise 3.4. Thus the set $f^{-1}(C)$ is in fact a closed subset of X.

We repeat the same argument using g and B. This gives that the sets $f^{-1}(C)$ and $g^{-1}(C)$ are both closed in X, hence the union $f^{-1}(C) \cup g^{-1}(C)$ is a closed set, again using exercise 3.4. Observe that $(f \cup g)^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$, so $(f \cup g)^{-1}(C)$ is closed. Thus for every $C \subset Y$ closed, $(f \cup g)^{-1}(C)$ is closed, so exercise 3.9 says that $(f \cup g)$ is continuous.

(b) There are many possible answers to this question. Here is one.

Let $X = [0, 1]$ with the usual topology as a subset of $\mathbb{R}$. Let $A = [0, \frac{1}{2}]$ and $B = (\frac{1}{2}, 1]$, so that $X = A \cup B$. Observe that B is not a closed subset of X - you've proved similar statements on earlier homeworks. Let $f : A \to [0, 1]$ be constant with value 0, and let $g : B \to [0, 1]$ be constant function with value 1. We are again using the standard topology on the target space $[0, 1]$. Recall from a previous homework that constant functions are always continuous. Note that $A \cap B = \emptyset$, so we can define the function $(f \cup g)$.

Then consider the open set $[0, 1/2) \subset [0, 1]$ (Recall that $[0, 1]$ has the subspace topology). The preimage of this set under $(f \cup g)$ is $(f \cup g)^{-1}([0, 1/2)) = A = [0, 1/2]$, which is not open in $[0, 1]$ - $1/2$ is not interior. Thus the function $(f \cup g)$ fails to be continuous.

Remarks: Many peoples' proofs in part (1) never made use of the fact that A and B are closed. But we can see from part (2) that any proof that leaves this fact out must be incorrect. Otherwise that proof would imply that the function $(f \cup g)$ we described in part (b) above would be continuous. When writing proofs, it's often useful to check where you use each of the hypotheses of the theorem. If you didn't use them all, you might have made a mistake in your proof. (Of course, this doesn't necessarily mean you've made a mistake - sometimes a theorem may be written with extraneous hypotheses. In this case, though, we know that the condition that A and B are closed is necessary).

Question 2: Show that $(0, 1), (0, \infty)$, and $(-\infty, \infty)$ are all homeomorphic.

Solution: We recall that because homeomorphism is an equivalence relation, it suffices to show that any two pairs from the above are homeomorphic, and the transitive property gives that the final pair is homeomorphic.

We begin by showing that $(0, \infty)$ and $(0, 1)$ are homeomorphic. Let $f : (0, 1) \to (0, \infty)$ be given by $f(x) = \frac{1}{x} - 1$. First, we claim that this function is injective on its domain. Assume that we have $x, y \in (0, 1)$ such that $f(x) = f(y)$. Then $\frac{1}{x} - 1 = \frac{1}{y} - 1$, which after some algebra yields $x = y$, so that f is injective. We next check surjectivity. Let $y \in (0, \infty)$ be given. We claim that $\frac{1}{y+1}$ is a preimage for y under f. Observe that since $y > 0$, we have that $y + 1 > 1$, hence $\frac{1}{y+1} < 1$. and also $\frac{1}{y+1} > 0$, so that this is indeed in the domain of the function. We then observe

$$f\left(\frac{1}{y+1}\right) = \frac{1}{\frac{1}{y+1}} - 1 = y + 1 - 1 = y$$

so that f is surjective. We therefore have that f is a bijective, and the inverse function $f^{-1} : (0, \infty) \to (0, 1)$ is given by $f^{-1}(x) = \frac{1}{x+1}$.

To show that f is a homeomorphism, it remains to show that both f and f^{-1} are continuous. We begin by showing that f is continuous. We can use the $\delta - \epsilon$ definition of continuity from Theorem 2.17 in the text. To use this, we fix $x \in (0, 1)$, and let $\epsilon > 0$ be given. We can choose $\delta > 0$ such that $\delta < \min\{1 - x, x/2, \epsilon x^2/2\}$. Note that our choice of δ depends only on x and ϵ.

This gives three inequalities. The first two are that $\delta < 1 - x$ and $\delta < x/2$. Together, these guarantee that the interval $(x - \delta, x + \delta)$ is contained in $(0, 1)$. We also have that if $|x - y| < \delta$, then $x - \delta < y$. Since $\delta < x/2$, we have

$$\frac{x}{2} = x - \frac{x}{2} < x - \delta < y \Leftrightarrow \frac{1}{y} < \frac{2}{x} \tag{$\dagger$}$$

Our choice of δ also gives the the third inequality $\delta < \epsilon x^2/2$. Thus, if $|x - y| < \delta$, we have $|x - y| < \epsilon x^2/2$, so that if $|x - y| < \delta$,

$$|f(x) - f(y)| = \left|\left(\frac{1}{x} - 1\right) - \left(\frac{1}{y} - 1\right)\right| = \left|\frac{y - x}{xy}\right| = \frac{|y - x|}{xy} < \frac{|y - x|}{x} \cdot \frac{2}{x} < \frac{\epsilon x^2/2 \cdot 2}{x^2} = \epsilon$$

as desired. Note that we use the inequality ($\dagger$) and the fact that both x and y are positive. This establishes that f is continuous.

We then use Theorem 2.17 to show that f^{-1} is continuous as well. Fix $x \in (0, \infty)$ and $\epsilon > 0$. Choose $\delta < \min\{x/2, \epsilon\}$.

We choose $\delta < x/2$ so that the interval $(x - \delta, x + \delta)$ is contained in the domain $(0, \infty)$. Our choice of δ also guarantees that $\delta < \epsilon$. Note also that $\frac{1}{y+1} < x + 1$. This follows from the inequality $y > 0$, which after adding 1 to both sides and inverting yields $\frac{1}{y+1} < 1$, and the fact that since $x > 0$, we have $1 < x + 1$. Combing these gives $\frac{1}{y+1} < 1 < x + 1$, which is the desired inequality.

Then assuming that $|x - y| < \delta$, hence $|x - y| < \epsilon$, we have

$$|f(x) - f(y)| = \left|\frac{1}{x+1} - \frac{1}{y+1}\right| = \left|\frac{(y+1) - (x+1)}{(x+1)(y+1)}\right| = \frac{|y - x|}{(x+1)} \cdot \frac{1}{y+1} < \frac{|y - x|}{(x+1)} \cdot (x+1) = |x - y| < \epsilon$$

This establishes that f^{-1} is continuous.

We use a similar process to show that $(-\infty, \infty)$ and $(0, \infty)$ are homeomorphic, although we won't write the full details here for brevity. One possible homeomorphism you could choose is the function $g : (-\infty, \infty) \to (0, \infty)$ given by $g(x) = e^x$, which has inverse $g^{-1}(x) = \log x$ the natural logarithm. Both g and g^{-1} are continuous, again using the $\delta - \epsilon$ definition of continuity. (In fact, this is almost a tautology, as e^x is defined for irrational numbers precisely so that the function will be continuous). This gives a homeomorphism between $(-\infty, \infty)$ and $(0, \infty)$.

Question 2: Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be continuous. Let $X \subset \mathbb{R}^n$ be any subset, and let $g : X \to \mathbb{R}^n$ be the restriction $g = f|_X$. Show that g is continuous.

Solution: The topology on X is the subspace topology. The open sets of X are exactly the sets of the form $U \cap X$, where U is an open subset of $\mathbb{R}^n$.

Let $U \subset \mathbb{R}^n$ be an open set. Then

$$\begin{aligned}
g^{-1}(U) &= \{x \in X : g(x) \in U\} \\
&= \{x \in X : f(x) \in U\} \\
&= \{x \in \mathbb{R}^n : f(x) \in U\} \cap X \\
&= f^{-1}(U) \cap X
\end{aligned}$$

The second line above follows because by definition $g(x) = f(x)$ for $x \in X$.

Observe that $f^{-1}(U)$ is open in $\mathbb{R}^n$, since $U \subset \mathbb{R}^n$ is open and f is assumed continuous. Therefore, the set $g^{-1}(U)$ is the intersection of an open set in $\mathbb{R}^n$ and the set X, hence $g^{-1}(U)$ is open in

X. Thus for every open $U \subset \mathbb{R}^n$, we have $g^{-1}(U)$ is open in X, and therefore g is continuous.

Question 3: Let $f : X \to Y$ be a function between topological spaces. Show that f is continuous when either (a) X has the discrete topology, or (b) Y has the indiscrete topology.

Solution: (a) Let X have the discrete topology. This means that *all* subsets of X are open sets. Then let $U \subset Y$ be an open set. The set $f^{-1}(U)$ is open in X since all subsets of X are open. Therefore, for every $U \subset Y$ open, $f^{-1}(U) \subset X$ is open, that is, the function f is continuous.

(b) Let Y have the discrete topology. This means that the only open subsets of Y are Y itself and $\emptyset$. Then let $U \subset Y$ be an open set. There are two cases. In the first case, we have $U = Y$, so that $f^{-1}(Y) = X$ is open in X since any topology on X contains X. In the second case, we have $U = \emptyset$, so that $f^{-1}(\emptyset) = \emptyset$, which is open since any topology on X contains $\emptyset$. Thus for any $U \subset Y$ open, we have $f^{-1}(U) \subset X$ is open, and therefore f is continuous.

Question 4: Let $Y = \{a, b, c\}$ with the topology $\{\emptyset, \{a\}, \{a, b\}, \{a, b, c\}\}$. Give an example of a non-constant, continuous function $f : [0, 1] \to Y$, where $[0, 1] \subset \mathbb{R}$ has the usual topology. Will the function remain continuous if you instead take the discrete topology on Y. Why?

Solution: There are many possible answers to this question. One possible answer -

Define $f : [0, 1] \to Y$ by

$$f(x) = \begin{cases} a & 0 \leq x < \frac{1}{2} \\ c & \frac{1}{2} \leq x \leq 1 \end{cases}$$

Clearly f is non-constant, as it takes two different values. To observe that it is continuous, we consider the preimages of the open sets of Y -

$$\begin{aligned} f^{-1}(\{a, b, c\}) &= [0, 1] \\ f^{-1}(\{a, b\}) &= [0, 1/2) \\ f^{-1}(\{a\}) &= [0, 1/2) \\ f^{-1}(\emptyset) &= \emptyset \end{aligned}$$

Observe that each of these preimages is an open set in $[0, 1]$, the first and last by the definition of all topologies and the last because $[0, 1/2) = (-\infty, 1/2) \cap [0, 1]$ and $(-\infty, 1/2)$ is open in $\mathbb{R}$ (the standard topology on $[0, 1]$ is the subspace topology inherited from $\mathbb{R}$.). Since the preimage of every open set in Y is an open set in $[0, 1]$, the function is continuous.

The function will not remain continuous if the topology on Y is the discrete topology. If Y has the discrete topology, then $\{c\}$ is an open set in Y, and $f^{-1}(\{c\}) = [1/2, 1]$ is not an open set in $[0, 1]$, as $1/2$ is not an interior point.

Remark: One can prove that *any* non-constant function from $[0, 1]$ with the standard topology to any topological space with the discrete topology cannot be continuous using connectedness - see Section 3.2 and Theorem 3.15 in the text.