MAT 364 Topology

Problem Set 8

Solutions

1. (a) We need to check that the complement of $\bar{D}a, r$, ie the set $U = \{x \in X : d(a, x) > r_0\}$ is open. Pick $x \in U$; to show that x is interior, find a neighborhood of x contained in U. Suppose that $d(x, a) = R$ (note that $R > r$), and take a positive $\epsilon < R - r$. Now, $D(x, \epsilon) \subset U$: indeed, if $y \in D(x, \epsilon)$, then $d(x, y) < \epsilon$, and by the triangle inequality $d(a, y) \geq d(a, x) - d(x, y) > R - \epsilon > r$, so $y \in U$.
(b) Consider the metric d on X such that $d(x, x) = 0$, and $d(x, y) = 1$ when $x \neq y$. We know that this metric induces discrete topology, thus every set is closed. In particular, $D(a, 1)$ is closed. (We can also notice that $D(a, 1)$ consists of a single point a, and check that this is closed.) By contrast, the set $\bar{D}(a, 1)$ equals to the whole space X, since $d(a, x) \leq 1$ for every point x.
2. (a) Fix some point $x_0 \in X$, and consider open balls $D(x_0, n)$, where $n = 1, 2, 3, \ldots$. Clearly, these form an open cover for the whole space X, and thus for A: every point has a finite distance to x_0, and thus will be in $D(x_0, n)$ when n is large. Since A is compact, we should be able to find a finite subcover. If $D(x_0, N)$ is the largest ball in the subcover, all the other balls are contained in it (they are concentric), and it follows that $d(x, a) < N$ for all $x \in A$. So A is bounded.
(b) Following the hint, suppose that A is not closed, and then there exists some point $a \notin A$ which is not exterior. Note that every set of the form $X - \bar{D}(a, r)$ is open. (See Problem 1.) The union of all such sets (when r ranges over all positive numbers) is $X - \{a\}$, so (since $a \notin A$) we get an open cover for A. By compactness of A, there is a finite subcover; let r_0 be the smallest value for $X - \bar{D}(a, r)$ in the subcover. Then the union of all the sets in the subcover is $X - \bar{D}(a, r_0)$ (again, the balls $\bar{D}(a, r_0)$ are concentric; draw a picture to make this clearer); it follows that A is contained in $X - \bar{D}(a, r_0)$, thus $D(a, r_0)$ is disjoint from A, and a is exterior for A. Contradiction.
3. The two topologies are the same. To show this, we need to check that every set that is open in the metric d is also open in standard topology, and vice versa. We know (see previous hw) that an open ball $D(a, r)$ for the metric d looks like a rhombus centered at a, with diagonals of length $2r$. Thus, a set U is open with respect to d if every point x of U is contained in U together with a small rhombus centered at x. But any such rhombus necessarily contains a round disk centered at x; thus, U is open in the standard Euclidean topology. Conversely, if U is open in the Euclidean topology, every point comes with a round disk around it, and since a dick contains a rhombus, U is open in our metric.