MAT 364 Topology

Problem Set 7

due Wednesday, October 27

Problem 1. Let X be a topological space. Suppose that Y is a subset of X. We can define a topology on Y as follows. If $\mathcal{T}_X = \{U_i\}$ is the topology on X, let $\mathcal{T}_Y$ to be the collection of all sets $U_i \cap Y$. (Notice that $U_i \cap Y$ are subsets of Y.)
Check that $\mathcal{T}_Y$ is indeed a topology on Y (ie it satisfies the axioms). It is called a *subspace topology.*
Solution. First, notice that $Y = X \cap Y$ and $\emptyset = \emptyset \cap Y$ will be open. Next, if $A_i = U_i \cap Y$ are subsets in Y, then $\cup_i A_i = (\cup_i U_i) \cap Y$, and $\cap_i A_i = (\cap_i U_i) \cap Y$ by De Morgan Laws. So if A_i's are open in Y, ie U_i's are open in X, then $\cup_i U_i$ is open in X, and so $\cup_i A_i$ is open in our new topology on Y. Finite intersections are similar.

Problem 2. Suppose that X is a topological space, Y is a subset of X. As explained in Problem 1, Y can be considered as a topological space (equipped with subspace topology). Prove that if X is compact, and Y is closed in X, then Y is also compact.
Solution. If Y is closed in X, then $X - Y$ is open. Now, let V_i's be some open sets (in the subspace topology on Y) that cover Y. From this, let's construct an open cover for X: for each V_i there is an open set U_i in X such that $V_i = U_x \cap Y$; if we take all U_s and the open set $X - Y$, we get an open cover for X. X is compact, so we can find a finite collection that still covers X and consists of some U_i's and perhaps $X - Y$. Taking intersection with Y, we find that the corresponding finite collection of V_i's covers Y.

Problem 3. Consider the set $\mathbb{R}^2$. For any two points $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$, define the distance $d(\mathbf{x}, \mathbf{y})$ by the formula

$$d(\mathbf{x}, \mathbf{y}) = |x_1 - y_1| + |x_2 - y_2|.$$

(a) Prove that d satisfies the axioms for a distance, ie $(\mathbb{R}^2, d)$ is a metric space.

(b) Sketch the unit disk $D(0, 1)$ centered at 0 for this metric.

Solution. (a) Clearly, $d(\mathbf{x}, \mathbf{y}) \geq 0$, $d(\mathbf{x}, \mathbf{y}) = 0$ iff $x = y$, and $d(\mathbf{y}, \mathbf{x}) = d(\mathbf{x}, \mathbf{y})$. To check the triangle equality for points $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2)$, $\mathbf{z} = (z_1, z_2)$ notice that $|x_1 - z_1| \leq |x_1 - y_1| + |y_1 - z_1|$. Add this to the corresponding inequality for the second coordinate to show that $|x_1 - z_1| + |x_2 - z_2| \leq (|x_1 - y_1| + |x_2 - y_2|) + (|y_1 - z_1| + |y_2 - z_2|)$.
(b) For this, solve $|x| + |y| \leq 1$. The answer is the rhombus with vertices at $(\pm 1, \pm 1)$.

Please also do Exercise 3.3 p. 40, Exercise 3.9 p. 43.

Solution for 3.9 Recall that A is open iff $X - A$ is closed. Now, if $f : X \to Y$ is continuous, ie f^{-1}(open) is open, and B is a closed set in Y, $Y - B$ is open and so $f^{-1}(Y - B)$ is open. But since $f^{-1}(Y - B) = X - f^{-1}(B)$, it follows that $f^{-1}(B)$ is closed. The converse (showing that the function has to be continuous if preimages of closed sets are closed) is similar.