MAT 364 Topology

Problem Set 6

due Wednesday, October 13

Problem 1. Determine whether the following sets are compact or not. Prove your answers.
(a) $X = \{(x, y) \in \mathbb{R}^2 | x \geq 0, y \geq 0\}$, the graph of the function $y = \sin x$.
Solution. Not compact since it's not bounded (x an be arbitrarily large). Here and below, we use the theorem that says that a set in $\mathbb{R}^n$ is compact if and only if it is bounded and closed.
(b) $Y = \{(x, y) \in \mathbb{R}^2 | x \geq 0, y \geq 0\}$, the first quadrant . **Solution.** Not compact since not bounded.
(c) $Z = \{x^2 + y^2 + z^2 = 1\}$, a sphere in $\mathbb{R}^3$. **Solution.** This is compact. Indeed, the sphere is bounded (it is contained in a ball of any bigger radius) and closed (check this by verifying that any point not on the sphere is interior).
(d) $V = \{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots\}$. **Solution.** V is both closed in $\mathbb{R}$ and bounded, therefore compact. "Bounded" is clear. To check "closed", verify that every point $x \notin V$ is exterior. If $x > 1$ or $x < 0$, this is clear; if x is between 0 and 1, consider the neighborhood whose radius is less than the distance from x to the nearest points of the form $1/n$.
(e) $W = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots\}$. **Solution.** This set is not closed, and therefore, non-compact. Indeed, $0 \notin V$is not an exterior point, because every neighborhood of 0 will contain some point $1/n \in V$.
(d) $\mathbb{Q} \cap [0, 1]$, the set of rational numbers q such that $0 \leq q \leq 1$. **Solution.** This set is not closed, therefore non-compact. Every irrational point has rationals in every neighborhood, and thus cannot be exterior.

Problem 2. Suppose $A, B \subset \mathbb{R}^n$ are two compact sets. Prove that $A \cup B$ is compact. For more practice, please give three different proofs:
(a) from the definition with convergent subsequences
(b) from the characterization of compact sets in $\mathbb{R}^n$
(c) from the definition with open covers (to be discussed in Friday class).
Solution. (a) Consider a sequence $\{x_n\} \in A \cup B$. We'd like to find a convergent subsequence. For this, break $\{x_n\}$ into two sequences, the one consisting of points x_n lying in A, the other of points x_n lying in B. At least one of these will actually be an infinite sequence. (Otherwise we only have finitely many x_n's, a contradiction.) Now, if there is an infinite sequence in A, we can find a convergent subsequence (A is compact); this is a convergent subsequence of the original sequence $\{x_n\}$.
(b) We know that both A and B are closed and bounded, and need to check the same for $A \cup B$. Indeed, if A is in the ball $D(0, r_A)$, B is in the ball

$D(0, r_B)$, let R to be the biggest of r_A, r_B, and then $A \cup B$ is in the ball $D(0, R)$. For "closed", recall that the union of two closed sets is always closed.
(c) Given a cover for $A \cup B$ by open sets, we want to get rid of some of the sets, so that only finitely many remain and $A \cup B$ is still covered. Notice that the cover for $A \cup B$ covers A, so we can pick finitely many sets of that cover to cover A. Similarly, find a finite subcover for B. Now, put the two finite subcovers for A and B together; this is a finite subcover for $A \cup B$.

Problem 3. We will prove in class on Friday that if $f : D \to R$ is a continuous function, and X is a compact set in D, then $f(X)$ is compact. (Here $D \subset \mathbb{R}^n$, $R \subset \mathbb{R}^m$ as usual). Show that
(a) if $f : D \to R$ is continuous, and X is closed in $\mathbb{R}^n$, $X \subset D$, then $f(X)$ doesn't have to be closed.
(b) if $f : D \to R$ is continuous, and X is a bounded subset of D, then $f(X)$ doesn't have to be bounded.
(Give counterexamples and justify all the properties you need.)
Solution. (a) Consider the function $f = 1/x$, $X = [1, +\infty)$ a closed set. $f : (0, +\infty) \to \mathbb{R}$ is continuous, but $f(X) = (0, 1]$ is not closed.
(b) Same function, $f : (0, +\infty) \to \mathbb{R}$, $f = 1/x$. Take bounded set $X = (0, 1)$, then $f(X) = (1, +\infty)$ is unbounded.