MAT 364 Topology

Problem Set 5

due Wednesday, October 6

Problem 1. Determine whether the following sets are connected or not. Prove your answers.
(a) $X =$ the union of the x-axis and the y-axis in the (x, y)-plane $\mathbb{R}^2$
(b) $Y =$ the union of the z-axis and the circle $\{x^2 + y^2 = 1, z = 0\}$ in the 3-space $\mathbb{R}^3$ (with coordinates (x, y, z))
(c) $Z = \{x^2 + y^2 + z^2 = 1\}$, ie a sphere in $\mathbb{R}^3$.
Solution. (a) and (c) are connected. To prove this, use a theorem we discussed in class: a union of connected sets will be connected if all the sets share a point. For (a), we have the union of two intersecting lines (each is connected as we proved in class). For (c), the sphere can be represented as a union of meridians (each of which is homeomorphic to a closed interval and therefore connected); the meridians all go through the North pole (and the South pole), thus the union is connected.
(b) Y is disconnected, which is easy to see informally, because we have two disjoint pieces, the line and the circle. To make a proof out of this observation, check that each of the pieces is open in Y. (Note that they are relatively open, but not open in $\mathbb{R}^3$.)

Problem 2. Suppose X is connected.
(a) Does $IntX$ have to be connected?
(b) Does ClX have to be connected?
Prove or give counterexamples.
Solution. (a) $IntX$ can be disconnected. Example: take two disjoint open disks in the plane and connect them by a line segment. (For instance, take disks or radius 1 centered at (0,0) and (3,0), and connect by a segment of the x-axis.) The interior will be just the two disjoint disks since the line segment has no interior (check!), and is thus disconnected (check!).
(b) If X is connected, ClX is connected. Indeed, suppose not – suppose that $ClX = A \cup B$, where A, B are disjoint non-empty open sets. Since $X \cap A$ and $X \cap B$ would then be relatively open on X (why?), they are disjoint, and we have $X = (X \cap A) \cup (X \cap B)$ because $X \subset ClX$, we have to conclude that $X \cap A$ or $X \cap B$ is empty (otherwise X would not be connected). Suppose $X \cap B = \emptyset$, then $X \subset A$. Now, pick any point $b \in B$. Since $b \in ClX$ is a limit point for X, every neighborhood of b contains points of X, and thus points of A. However, since $B$is open, b has a neighborhood that's contained entirely in B (and has no points from A since A and B are disjoint). Contradiction.

Problem 3. (a) Show that $\mathbb{R}$ is homeomorphic to $(0, +\infty)$.

(b) Show that $\mathbb{R}$ is *not* homeomorphic to $[0, +\infty)$.

Solution. (a) A homeomorphism is given by the continuous bijection $f(x) = e^x$ (whose inverse $\ln x$ is also continuous).

(b) This is similar to proving that $[a, b]$ and (a, b) are not homeomorphic (we did this in class). By contradiction, suppose that $f : [0, +\infty) \to \mathbb{R}$ is a homeomorphism. Let $c = f(0)$. Since f is one-to-one, nothing else maps to c, and if we take 0 out of $(0, \infty)$ and c out of $\mathbb{R}$, $f : (0, +\infty) \to (-\infty, c) \cup (c, +\infty)$ will be a homeomorphism. But this is impossible, since $(0, +\infty)$ is connected and $(-\infty, c) \cup (c, +\infty)$ is not.

Problem 4. Prove that a circle is not homeomorphic (a) to an open interval, (b) to a closed interval. This uses the same idea as Problem 3b above.

(a) is almost literally the same: if there were a homeomorphism from circle to an open interval, taking out one point we'd get a homeomorphism between a circle without a point and an open interval without a point. Circle without a point is homeomorphic to an open interval and is therefore connected; an interval without a point breaks into two disjoint open intervals and is disconnected.

(b) USing the same idea, if there were such a homeomorphism f, we'd get a homeomorphism between a circle without some point a and a closed interval without the corresponding point $f(a)$. Now unless $f(a)$ happens to be an endpoint of the closed interval, the interval becomes disconnected after $f(a)$ is taken out, and we can argue as before. If $f(a)$ is the endpoint, we'll just have to put a back and pick a different point in the circle (the one that doesn't go to any of the endpoints).