MAT 364 Topology

Problem Set 2

Solutions

2.7. $Fr(A) = Fr(\mathbb{R}^n - A)$.

Proof. By definition, frontier of the set consists of all points whose every neighborhood contains points from both the set and its complement. Thus, $Fr(A)$ consists of points whose every neighborhood contains points of A and points of $\mathbb{R}^n - A$. Similarly, $Fr(\mathbb{R}^n - A)$ consists of points whose every neighborhood contains points of $\mathbb{R}^n - A$ (the set itself) and $\mathbb{R}^n - (\mathbb{R}^n - A)$ (its complement). Since $\mathbb{R}^n - (\mathbb{R}^n - A) = A$, we conclude that $Fr(A)$ and $Fr(\mathbb{R}^n - A)$ consist of exactly the same points. $\square$

2.8. $Fr(A) = Cl(A) \cap Cl(\mathbb{R}^n - A)$.

Proof. By definition, $Cl(A) = A \cup Fr(A)$, and $Cl(\mathbb{R}^n - A) = (\mathbb{R}^n - A) \cup Fr(\mathbb{R}^n - A)$. Now, consider the intersection of $A \cup Fr(A)$ and $(\mathbb{R}^n - A) \cup Fr(\mathbb{R}^n - A)$. Since no point can be in A and $\mathbb{R}^n - A$ at the same time, the intersection equals to the union of the sets $Fr(A) \cap (\mathbb{R}^n - A)$, $A \cap Fr(\mathbb{R}^n - A)$, and $Fr(A) \cap Fr(\mathbb{R}^n - A)$. Now use the preceding exercise: $Fr(A) = Fr(\mathbb{R}^n - A)$, and thus

$$Cl(A) \cap Cl(\mathbb{R}^n - A) = (A \cap Fr(A)) \cup ((\mathbb{R}^n - A) \cap FrA) \cup Fr(A)).$$

The set on right hand side of this formula clearly contains $Fr(A)$ (because of taking the last union) and is contained in $Fr(A)$ (because all the three sets in the union are contained in $Fr(A)$). Thus, $Cl(A) \cap Cl(\mathbb{R}^n - A) = Fr(A)$. $\square$

2.15. If A, B are closed, then $A \cap B$ and $A \cup B$ are closed.

Proof. We proved a similar statement for open sets in class. This can be proved directly by a similar argument, or derived by looking at complements. Indeed, if A, B are closed, then $\mathbb{R}^n - A$ and $\mathbb{R}^n - B$ are open, and so (as we proved) $(\mathbb{R}^n - A) \cup (\mathbb{R}^n - B)$ and $(\mathbb{R}^n - A) \cap (\mathbb{R}^n - B)$ are open. But set theory tells us that $(\mathbb{R}^n - A) \cup (\mathbb{R}^n - B) = \mathbb{R}^n - (A \cap B)$, and $(\mathbb{R}^n - A) \cap (\mathbb{R}^n - B) = \mathbb{R}^n - (A \cup B)$, so $A \cap B$ and $A \cup B$ are closed because their complements are open. (This is by the exercises 2.10-2.11 that we proved in class.)

Give an example of an infinite union of closed sets which is not closed. Solution: one can take, for example, the union of all closed intervals of the form $[1/n, 1 - 1/n]$. This union equals $(0, 1)$, which is not a closed set. Another example would be a union of one-point sets $\{1/n\}$. (Check that this is not closed – 0 is not an exterior point.) $\square$

2.16. Prove that an infinite intersection of closed sets is closed.

Proof. Use the fact we proved in class: infinite union of open sets is open, and pass to the complements as in 2.15. $\square$