MAT 364 Topology

Problem Set 1

Solutions

2.6. $Fr(A)$ is closed for any set A.

Proof. By definition of a closed set, we need to show that every point $x \notin Fr(A)$ is exterior for $Fr(A)$. Indeed, pick $x \notin Fr(A)$. We know that for any point in $\mathbb{R}^n$, exactly one possibility holds with respect to A: 1) it's an interior point for A, 2) it's an exterior point for A, 3) it's a frontier point for A. Since $x \notin A$, x must be an interior or an exterior point for A. 1) Suppose x is interior for A. This means that there's a neighborhood $D(x, r)$ of x contained in A. We'll show that this whole neighborhood is in fact contained in $IntA$; then it follows that $D(x, r)$ is disjoint from $Fr(A)$, which implies that x is exterior for $Fr(A)$. To see that $D(x, r) \subset Int(A)$, notice that any point $y \in D(x, r)$ has a small neighborhood $D(y, r')$ that is contained inside $D(x, r) \subset A$. (Draw the disk $D(x, r)$ and a smaller disk around y which is contained in the large disk. We did this many times in class.) This implies that y is interior for A, and so $D(x, r)$ does not intersect $Fr(A)$. 2) The case x exterior for A is very similar: find a neighborhood of x lying outside of A, show that every point of that neighborhood is exterior for A, thus the whole neighborhood is disjoint from $Fr(A)$, and so x is exterior for $Fr(A)$. □

This proof is somewhat similar to the proof of Thm 2.9. In fact this question can be derived from Thm 2.9, the identity $Cl(A) = Fr(A) \cup Int(A)$, and the fact that $Int(A)$ is open.

2.9. A open iff $A = Int(A)$.

Proof. Suppose A is open. By definition, this means that every point of A is interior, ie $A \subset Int(A)$. The other inclusion, $Int(A) \subset A$, is always true (since interior point has to lie in A together with a neighborhood). Therefore, $A = Int(A)$.
Conversely, suppose $A = Int(A)$. We can conclude that A is open if we show that the interior of every set is open. (Because then $Int(A)$ is open.) Indeed, if a point y is in $Int(A)$, it has a disk neighborhood $D(y, r)$ contained in A, and for any other point in this disk we can find a smaller disk contained in $D(y, r)$, and thus in A; therefore, every point of the disk $D(y, r)$ lies in $Int(A)$, and so y is interior for $Int(A)$. (Compare with the argument in 2.6 above.) □