A BRIEF INTRODUCTION TO LOBACHEVSKIAN GEOMETRY

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Abstract. These are lecture notes for MAT 360, Spring 2011 and Fall 2011.

1. A bit of history and discussion of axiomatics

Euclid's famous "5th postulate" states that, given a line l in the plane and a point A not on l, there exists a *unique* line passing through A and parallel to l. The point of the postulate is that we can never have two different lines with this property. Indeed, it is not hard to prove (using other axioms) that a line through A that is parallel to l exists; such a line can be constructed with a compass and straightedge.

What exactly does it mean that the 5th postulate cannot be derived from other axioms? People certainly tried to prove the uniqueness of the parallel line - the statement looks like it could be a theorem. (Recall, for example. that uniqueness of a perpendicular from a given point to the given line can be easily proved.) However, no-one was able to prove this uniqueness of parallels. Does this mean that it cannot be proved? Not really: perhaps the proof was just too difficult and elusive for anyone to discover. To demostrate that the 5th postulate cannot be proved, one really needs to show that it is "independent" from the other axioms; indeed, one needs to demonstrate that there exists a "geometry" that satisfies all the other axioms (such as, through any two points there is a unique line, the existence of certain isometries, etc) but not the uniqueness of parallels. Such a "geometry" was first constructed by Lobachevsky in early 19th century; the construction he came up with is now called the Lobachevskian plane.

Before delving into the details of the construction, let us discuss what is meant by "geometry" and by the lack of uniqueness of parallels. Does it mean that in an alternative universe, one could suddenly draw a bunch of different lines through A, and none of those lines would intersect l? In a sense, yes – but the "lines" and "points" might look quite different from those in Euclidean geometry. To understand this better, let us consider the axioms of the arithmetics of numbers. Those are very familiar - one can add and multiply integer numbers, and they satisfy certain properties, such as $a+b = b+a$, $a(b+c) = ab+ac$, $(a+b)c = ac+bc$, $a+(-a) = 0$, $ab = ba$, etc. If one accepts the rest of the properties as axioms, can the last one, $ab = ba$, be proved as a theorem? To show that this property is independent from other axioms, one needs to construct some alternative version of "numbers" that satisfy all the other axioms, but not $ab = ba$. Again, this doesn't mean that in alternative universe, the multiplication table goes wrong, and $5 \cdot 3 \neq 3 \cdot 5$; rather, "numbers" can be some more involved objects. For example, if the inhabitants of the alternative universe are working with matrices instead of numbers, this will be exactly the situation described above. (Recall that $n \times n$ matrices can be added and multiplied, and have a lot of nice properties, such as $A + B = B + A$, etc; however, in general AB and BA are different matrices.)