MAT 360 Geometry

Homework 8

due Thursday, Nov 3

Problem 1. Prove that any isometry I is a bijective map (in the sense of MAT 200); you have to check that

(a) I injective, i.e. I always sends different points to different points; and

(b) I surjective, i.e. its image is the whole plane.

Part (a) is easy to prove from the definition of isometry as a distance-preserving map. For part (b), a proof from scratch is harder; for an easier approach, use some of the theorems about isometries we developed so far.

Problem 2. Prove that a composition of two translations is a translation. What is the vector of translation for the composition?

Problem 3. Prove that any two congruent segments can be superimposed by some rotation. **Hint:** think two reflections.

Problem 4. Let l_1 and l_2 be two lines that intersect at the point O; R_1, R_2 denote reflections about these lines. In class, we discussed that the composition of these reflections is the same as the rotation about O by twice the angle between the lines. This may cause some confusion: indeed, you have to pay attention to the order of reflections (which one is performed first), to the direction of rotation (clockwise or counterclockwise), and to the choice of the angle (the lines cut the plane into 4 angles). We saw that, more precisely,

$$R_2 \circ R_1 = Rot_{O,2\alpha}, \tag{1}$$

where in the composition R_1 is performed first, and α stands for the angle measured *from* the line l_1 to the line l_2 *counterclockwise*. By convention, α is always taken to be less than 180° (but it can be acute, right or obtuse). The purpose of this problem is for you to get used to all these subtleties.

Now, let R_x be the reflection about the x-axis, R_{xy} the reflection about the line $x = y$.

(a) Using the formula above, what do you get for $R_{xy} \circ R_x$? Prove your answer directly: check where a few "nice" points go, and use the fact that an isometry is determined by its effect on any 3 non-collinear points.

(b) Repeat part (a) for the composition $R_x \circ R_{xy}$.

(c) Prove that $R_x \circ R_{xy}$ is the *inverse* of $R_{xy} \circ R_x$ (in the sense of MAT 200), i.e. $R_x \circ R_{xy}$ undoes the effect of $R_{xy} \circ R_x$. Therefore, if the first composition is a *counterclockwise* rotation by a certain angle, the second must be the *clockwise* rotation by the same angle. How does this fit with formula (1), where we only consider counterclockwise rotations? Explain.

Problem 5. Prove that a composition of a rotation and a translation is a rotation. (This is true for any order of the rotation/translation; for concretess, you can assume that the rotation is performed first.)

We will talk about a useful trick on Tuesday, you may wish to postpone doing this question until then.