MAT 341
Final Exam Checklist

The focus of MAT341 is on partial differential equations and on common problems arising from physics. The following topics will be on the test:

Terminology: Recognize heat/wave/potential problems as well Dirichlet/Neumann problems. For each problem, be able to state the equations and appropriate boundary conditions/initial conditions. This includes problems in infinite domains where the solution is required to be bounded.

Fourier series and Fourier integrals:

The formula for the Fourier series of a periodic function f with period $2a$ will be given (see the reference sheet) but you should know how to use them. Know how to derive and use formulas for the sine/cosine series for odd/even functions.

While you will not have to compute long and difficult integrals on the test, you should be able to use basic integration techniques (such as integration by parts). You should also be very familiar with odd and even functions (properties, graphs, integration). You should also know that Fourier series for linear combinations of sines and cosines (with matching periods!) such as $3\sin \pi x - 2\cos 7\pi x + \sin 4\pi x$ are given by these combinations themselves.

You should know convergence theorems (Theorem from 1.3, Theorem 2 from 1.4 and Theorems 3,4,5 which follow) and have an intuitive understanding of uniform convergence vs convergence for every x individually. When working with extensions, always draw the periodic extension on the whole interval $(-\infty, \infty)$ and be careful with endpoints of the given interval, otherwise you'll miss the possible jumps there.

Questions may include:

Computing Fourier series:

- Determine which of the given functions are periodic, find period
- Sketch graphs of the periodic extenstion of a function given on $(-a, a)$, and odd/even periodic extensions of a function on $(0, a)$
- Compute certain Fourier coefficients for a given function; write an explicit expression for a given Fourier coefficient but do not compute[1]
- compute the Fourier series for a periodic function or for a periodic extension of a function given on an interval $(-a, a)$
- Use odd/even extensions to find sine/cosine series for a given function on $(0, a)$

Convergence of Fourier series:

- determine to what value the Fourier series of the function converges at a given point; explain your answer.
- determine whether the convergence is uniform; explain your answer.

Fourier integral :

- Find Fourier integral representation of a function given on $-\infty, \infty)$ (the general formula will be on reference sheet).
- Find the Fourier sine and cosine integrals for a function given on $-\infty, \infty)$. State what extensions of the functions are used to find these representations; sketch their graphs.

Practice: 1.1 questions 1, 2; 1.2 questions 1, 5, 7, 10; 1.3 questions 2, 3, 5, 6; 1.4 questions 1, 3; p.118 questions 1, 7cde, 10, 11, 12, 13, 14, 15, 30, 30af; 1.9 questions 1, 2, 5.

The heat equation:

You should be familiar with the setup of the heat problem (differential equation itself, boundary conditions, intitial conditions), with all the steps required to solve it (see summary at the end of 2.5), and all the terminology (steady-state, transient solutions). It's important to remember that the equation for the transient solution should be *homogeneous*, so if the original equation or boundary conditions are non-homogeneous (for example, there's a constant added to the equation), the extra terms will probably cancel.

[1] make it as explicit as possible without computing: for example, if the function f is given by $f(x) = x^2$ for $0 < x \leq 1$, $f(x) = x$ for $1 < x < 2$, and you are working with $\int_0^2 f(x) \cos 3\pi x dx$, you should expand it as $\int_0^1 x^2 \cos 3\pi x dx + \int_1^2 x \cos 3\pi x dx$

Questions may include:

- Find the steady-state solution for a given equation with given boundary conditions. The equation may include extra terms such as generation; you will need basic MAT 303 skills to solve ordinary differential equations. Boundary conditions may be of different types.
- Explain the physical meaning of the steady-state solution (it's a "stable" solution after a lot of time has passed and the temperature distribution is not changing with time anymore).
- Find the equation and the boundary and initial conditions for the transient solution for a given problem (do not solve). Again, the equation may include extra terms for generation, and different boundary conditions may appear.
- Given a heat equation problem with boundary conditions and initial conditions, go through all the steps to solve the equation. You will *only* be required to solve the standard fixed end temps or insulated bar questions (2.3 or 2.4), with specific initial and boundary conditions. You should be able to produce the complete solution *without* the prompts for each step.
- Heat problems in infinite and semi-infinite rod (questions as above)

Practice: 2.2 questions 1–8; 2.3 questions 5–8; 2.4 questions 1–5; 2.10 questions 1, 3; 2.11 questions 1, 2; p. 205 questions 1-3, 5-7, 10, 11, 12, 14, 15.

Sturm-Liouville problem:

You should be familiar with the statement of the problem, the eigenvalues/eigenvectors terminology, and the orthogonality relation. Only the basic problem as in 2.7 equations (1)-(3) will be on the test (the more general Sturm-Liouville problem, equations (5)-(7) is not on the test).

Questions may include:

- recognize which of the given equations and boundary conditions are a Sturm-Liuoville problem.
- find eigenvalues and eigenvectors for a given Sturm-Liouville problem.
- state the orthogonality relation for the given problem.

Practice: 2.7 questions 2, 3, 4, p.209 28, 29 (28 and 29 have an extra x, but you should still be able to solve it).

The wave equation:

Questions may include:

- Find d'Alambert's solution (as a composition of two waves) for a given vibrating string problem with fixed ends. Find the relevant odd/even extensions for the functions given by the initial conditions. Illustrate the solution graphically, by averaging graphs as in 3.3. You should be able to go through this process with or without prompts at every step.
- Do similar work with d'Alembert solution for a semi-infinite string with fixed end. Be sure to note the difference between the finite case (periodic extensions of initial data are used) and infinite case (initial data is given on semi-infinite interval, odd/even extensions are used).
- Use separation of variables to write the general solution as a series. Use Fourier series to find the coefficients.
- State the eigenvalue problem associated to the given boundary value problem (we get the eigenvalue problem after the separation of variables). In some easy cases, solve the eigenvalue problem (or determine whether, for example, 0 is an eigenvalue, or whether there are positive/negative eignevalues). State orthogonality of eigenfunctions. Write the solution as a series in eigenfunctions.
- For non-homogeneous boundary value problem, write the solution as $u(x,t) = v(x) + w(x,t)$; find the function $v(x)$ and the equation/boundary conditions for $w(x,t)$.
- Describe the general behaviour of the solution; find frequencies of vibration.

Practice: 3.1 question 3; 3.2 questions 3, 4, 5, 6, 7, 9, 12, 13, 14; 3.3 questions 1-8; 3.6 questions 5,6; p.252-253 questions 1-5, 9-10.

The potential equation:

Questions may include:

- use separation of variables to solve the Laplace equation with appropriate boundary conditions (Dirichlet-type or Neumann-type as in 4.2, 4.3, or mixed type). You may be asked to solve the problem completely (in particular, compute the coefficients of any relevant Fourier series) or just to perform certain steps (for example, find eigenvalues).

- split a given problem into two problems (via $u = u_1 + u_2$) to make separation of variables applicable (you may be asked to do the splitting without solving the resulting problems); explain why the function u given as sum of the two solutions is indeed a solution for the original problem.

- solve the potential problem by finding a simple solution $v(x, y)$ that satisfies some of the boundary conditions, and reduce to homogenous boundary conditions on opposite sides (write $u(x, t) = v(x, u) + w(x, t)$ and obtain the equation for w). You may be asked to do the reduction without solving the resulting problem.

- Solve the Dirichlet problem in a disk (or half-disk or quarter-disk, with homogeneous boundary conditions on straight lines). The case at the end of 4.5 is not on the test. You may use solutions of the Cauchy-Euler equation without deriving them. You may be asked to solve the problem completely or just to perform several initial steps (such as separation of variables).

- Find the eigenvalue problem problem associated to a boundary value problem (as above) after the separation of variables; find eigenvalues/eigenfunctions for this problem.

- Similar questions for problems in infinite domains: find product solutions, set up the Fourier integral or Fourier series representing the solution, compute the integral/series in simple situations. For Fourier integrals, only the case of negative eigenvalues will appear on the test. (We saw some trickier questions in class with different kids of eigenvalues; this is not on the test.)

4.1 questions 1-4, 4.2 questions 5-9; 4.3 questions 1-2, 10; 4.4 questions 4, 5ab, 12-14 (assume that only negative eignevalues appear), 17-19; 4.5 questions 1-5; p.299-304 questions 1-9, 11-13, 25. Some of these questions require longer calculations than can be done on exam, but you should know how to approach each problem and be able to fully solve most of them.