MAT 320: FINAL EXAM

WEDNESDAY, DEC. 20

Your name:____
(please print)

No books, notes, or calculators. Unless a problem explicitly states "no explanation required", please try to write as detailed an explanation as possible. Explanations should be such that someone who does not know how to solve this problem (but knows all previous material) can follow your arguments and understand what you are doing; if in doubt, ask. When constructing examples of functions, you can simply sketch a graph (with explanation of its important features if needed); giving a formula is not necessary. **Answers without explanations will get very little partial credit!**

Notation:
$\mathbb{Z}$ — integer numbers
$\mathbb{N}$ — positive integers
$\mathbb{R}$ — real numbers

There are 10 problems in this exam. Each problem is worth 10 pts. You have 2 hours 30 minutes. Good luck!

	1	2	3	4	5	6	7	8	9	10	**Total**
Grade											

1. Determine whether each of the following statements is true or false, and circle your answer. No explanations or justifications are needed.

(a) If the function $f : X \to Y$ is surjective, then $f(A \cap B) = f(A) \cap f(B)$.

TRUE FALSE

(b) Let A be the set of all functions on $\mathbb{R}$ whose Taylor series about 0 converges for all x. The set A is countable.

TRUE FALSE

(c) A set of numbers $x \in [0, 1]$ whose decimal representation contains at least one "3" and at least one "5" is uncountable.

TRUE FALSE

(d) The sequence (x_n) is bounded if and only if every subsequence of (x_n) is bounded.

TRUE FALSE

(e) A continuous function $f : [0, +\infty) \to \mathbb{R}$ attains its maximum or its minimum.

TRUE FALSE

(f) If there exists $L \in \mathbb{R}$ such that $|f(x) - f(y)| \leq L|x - y|$ for all $x, y \in \mathbb{R}$, then the function f is uniformly continuous.

TRUE FALSE

(g) Suppose that for every $n \in \mathbb{N}$ there exists $\delta > 0$ such that

$$0 < |x + 1| < \delta \Longrightarrow |f(x)| < \frac{1}{n}.$$

It follows that $\lim_{x \to -1} f(x) = 0$.

TRUE FALSE

(h) If $f, g : (a, b) \to \mathbb{R}$ are differentiable, then $\lim_{x \to a+} \frac{f(x)}{g(x)} = \lim_{x \to a+} \frac{f'(x)}{g'(x)}$.

TRUE FALSE

(i) If $f : [a, b] \to \mathbb{R}$ is integrable, $f'(x)$ exists for all $x \in (a, b)$.

TRUE FALSE

(j) Every bounded function $f : [0, 1] \to \mathbb{R}$ is integrable.

TRUE FALSE

2. Let (x_n) be a sequence such that $\lim\limits_{x\to\infty} x_n = -\infty$.

(a) Show that (x_n) is bounded above.

(b) Show that there exists k such that $\sup\{x_n \mid x \in \mathbb{N}\} = x_k$.

3. Let (x_n) be a sequence such that $\lim_{x\to\infty} x_n = 0$.

(a) Consider the sequence (y_n), where

$$y_{2n-1} = x_n, \quad y_{2n} = \frac{1}{n} \qquad \text{for all } n \in \mathbb{N}.$$

(The first few terms of this sequence are $x_1, 1, x_2, \frac{1}{2}, x_3, \frac{1}{3}, \ldots$) Using the definition of the limit, prove that the sequence y_n converges and find its limit.

(b) Consider the sequence (z_n), where

$$z_{2n-1} = x_n, \quad z_{2n} = 1 - \frac{1}{n} \qquad \text{for all } n \in \mathbb{N}.$$

(The first few terms of this sequence are $x_1, 0, x_2, \frac{1}{2}, x_3, \frac{2}{3}, \ldots$) Does (z_n) converge? Justify your answer.

4. Let $x_1 = 2$ and $x_{n+1} = \dfrac{x_n}{1+x_n}$ for $n \geq 1$. Prove that the sequence (x_n) converges and find its limit.

5. (a) State the Intermediate Value Theorem.

(b) Let $f, g : [-1, 1] \to [-1, 1]$ be two continuous functions such that $f(-1) = -1$, $f(1) = 1$, $g(-1) = 1$, $g(1) = -1$.
Show that there exists $x \in [-1, 1]$ such that $(g \circ f)(x) = x$.

6. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with the property that $f(x) = 0$ whenever $|x| \geq 1$. Show that for any sequence $\{x_n\}$ in $\mathbb{R}$, the sequence $\{f(x_n)\}$ has a convergent subsequence.

7. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Suppose that f has n distinct roots. (A root of f is a number $x \in \mathbb{R}$ such that $f(x) = 0$.)
Show that $f'(x)$ has at least $n - 1$ roots.
Can $f'(x)$ have more than $n - 1$ roots?

8. (a) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Let $c \in \mathbb{R}$ be such that $f(c) < 0$. Using the definition of the continuous function, prove that there exists $\delta > 0$ such that $f(x) < 0$ for all $x \in (c - \delta, c + \delta)$.

(b) Let $g : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $g'(x)$ is continuous on $\mathbb{R}$. Let $c \in \mathbb{R}$ be such that $g'(c) < 0$. Prove that there exists $\delta > 0$ such that $g(x)$ is strictly decreasing on $(c - \delta, c + \delta)$.
(A function $g(x)$ is strictly decreasing if $g(x_1) < g(x_2)$ whenever $x_1 < x_2$.)

9. Consider the function

$$f(x) = \begin{cases} x^2, & x \in \mathbb{Q} \\ -x^2, & x \notin \mathbb{Q} \end{cases} .$$

At what points is $f(x)$ continuous?
At what points is $f(x)$ differentiable?

(You may use without proof the density of rational/irrational numbers, i.e. the fact that there exist sequences of rationals resp. irrationals converging to any given real number.)

10. The function $f : [0, 3] \to \mathbb{R}$ is defined as

$$f(x) = \begin{cases} -1 & x = 1 \\ 2 & x = 2 \\ 1 & \text{otherwise} \end{cases} \quad .$$

Is $f(x)$ integrable?

If so, find $\int_0^3 f(x)dx$.