MAT 319, Examples for Functions and Induction

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Here is a more rigorous proof of 1.1.9 and some more examples, which I did not have time to cover in class.

1.1.9 As shown in class, $h : \mathbb{R} \to \mathbb{R}$, $h(x) = (x+2)^2$.

(a) For $E := \{x \in \mathbb{R} : 1 \le x \le 2\} = [1,2]$, $h(E) = ?$

By drawing the graph, we get the idea that $h(E) = [9,16]$. A rigorous way to prove this is to use a proof of set equality. Use the definitions of direct image, E and h.

Claim: $h(E) = [9,16]$.

Pf: For any $y \in h(E)$, want to show $y \in h(E) \Leftrightarrow y \in [9,16]$.

$$\begin{array}{ll} & y \in h(E) \\ \Leftrightarrow & y = h(x), \text{ for some } x \in E, \text{ (by def. of direct image of } E) \\ \Leftrightarrow & y = (x+2)^2, \text{for some } x \in \mathbb{R}\ s.t.\ 1 \le x \le 2, \text{ (by def. of h,E)} \\ \Leftrightarrow & (1+2)^2 \le (x+2)^2 \le (2+2)^2 \\ \Leftrightarrow & 9 \le y \le 16 \\ \Leftrightarrow & h(x) \in [9,16]. \\ \therefore & h(E) = [9,16].\ \Box \end{array}$$

(b) For $G := \{x \in \mathbb{R} : 0 \le x \le 4\}, h^{-1}(G) = ?$

By looking at the graph of h, we see that $h^{-1}(G) = [-4,0]$.

Again, to prove this requires a proof of set equality. Use the definitions of inverse image, G, and h.

Claim: $h^{-1}(G) = [-4,0]$.

Pf: For any $x \in h^{-1}(G)$, want to show $x \in h^{-1}(G) \Leftrightarrow x \in [-4,0]$.

$$\begin{array}{ll} & x \in h^{-1}(G) \\ \Leftrightarrow & h(x) \in G, \text{ (by def. of inverse image)} \\ \Leftrightarrow & 0 \le h(x) \le 4, \text{ (by def. G)} \\ \Leftrightarrow & 0 \le (x+2)^2 \le 4 \text{ (by def. of h)} \\ \Leftrightarrow & 0 \le |x+2| \le 2 \\ \Leftrightarrow & 0 \le x+2 \le 2 \quad \text{or} \quad -2 \le x+2 \le 0, \text{ (by def. of abs. val.)} \\ \Leftrightarrow & -2 \le x \le 0 \quad \text{or} \quad -4 \le x \le -2 \\ \Leftrightarrow & x \in [-2,0] \cup [-4,-2], \text{ by def. of } \cup \\ \Leftrightarrow & x \in [-4,0], \text{ b/c } [-2,0] \cup [-4,-2] = [-4,0] \\ \therefore & h^{-1}(G) = [-4,0].\ \Box \end{array}$$

1.1.14 Show that $f(x) := \frac{x}{\sqrt{x^2+1}}$ is a bijection of $\mathbb{R}$ onto $\{y : -1 < y < 1\} = (-1, 1)$, i.e. show that:

$$f : \mathbb{R} \to (-1, 1) \text{ is a bijection}$$

Claim: $f : \mathbb{R} \to (-1, 1)$ is a bijection.

Pf: Want to show that (a) f is injective, and (b) f is surjective.

(a) For any $x_1, x_2 \in \mathbb{R}$, $x_1 \neq x_2$, want to show that $f(x_1) \neq f(x_2)$.
Proof by contradiction.
Suppose $f(x_1) = f(x_2)$. Then:

$$\begin{array}{rlcl} & \frac{x_1}{\sqrt{x_1^2+1}} & = & \frac{x_2}{\sqrt{x_2^2+1}} \\ \Rightarrow & \frac{x_1^2}{x_1^2+1} & = & \frac{x_2^2}{x_2^2+1} \\ \Rightarrow & (x_2^2+1)x_1^2 & = & (x_1^2+1)x_2^2 \\ \Rightarrow & x_2^2x_1^2 + x_1^2 & = & x_1^2x_2^2 + x_2^2 \\ \Rightarrow & x_1^2 & = & x_2^2 \\ \Rightarrow & x_1 & = & \pm x_2 \end{array}$$

But, $\frac{x_1}{\sqrt{x_1^2+1}} = \frac{x_2}{\sqrt{x_2^2+1}} \Rightarrow x_1 = x_2$. Contradiction! since $x_1 \neq x_2$, by assumption.
Thus, $f(x_1) \neq f(x_2)$.
$\therefore f$ is injective.
Note: One could do a direct proof using the other definition of injective, i.e. that $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

(b) By definition of surjectivity of $f : \mathbb{R} \to (-1, 1)$, Want to show that $R(f) = (-1, 1)$.
I'll do the "hard work" first and then refer to it.

$$\begin{array}{ll} & \forall x \in \mathbb{R}\ 0 \leq x^2 < x^2 + 1 \\ \Leftrightarrow & 0 \leq \frac{x^2}{x^2+1} < 1, \text{ (since } 0 < \mathrm{x}^2 + 1) \\ \Leftrightarrow & 0 \leq \left| \frac{x}{\sqrt{x^2+1}} \right| < 1 \\ \Leftrightarrow & 0 \leq \frac{x}{\sqrt{x^2+1}} < 1 \quad or \quad -1 < \frac{x}{\sqrt{x^2+1}} \leq 0, \text{ (by def. of abs. val.)} \\ \Leftrightarrow & \frac{x}{\sqrt{x^2+1}} \in (-1, 0] \cup [0, 1), \text{ (by def. of } \cup) \\ \Leftrightarrow & \frac{x}{\sqrt{x^2+1}} \in (-1, 1), \forall x \in \mathbb{R} \end{array}$$

So,

$$\begin{array}{ll} & y \in R(f) \\ \Leftrightarrow & \exists x \in \mathbb{R}\ y = f(x), \text{ by def. of R(f)} \\ \Leftrightarrow & y = \frac{x}{\sqrt{x^2+1}}, \text{ by def. of f} \\ \Leftrightarrow & y \in (-1, 1), \text{ as shown prev., } \frac{\mathrm{x}}{\sqrt{\mathrm{x}^2+1}} \in (-1, 1) \\ \Leftrightarrow & R(f) = (-1, 1). \\ \therefore & \text{f is surjective on } (-1, 1). \end{array}$$

Hence, f is injective and surjective $\Rightarrow$ f is bijective. $\square$

Monks with blue spots There are 100 polite monks living in a monastery, who have all taken a vow of silence. A bishop visits them and during the night paints a blue spot on the bald heads of 10 of the monks, none of whom are awoken by such deviancy. The next day, during his sermon, the bishop tells the monks:

- At least one of you has a blue spot on your head.
- Once one of you realizes that you realizes that you have a blue spot on your head, you must leave that night.

What happens?

Answer: By induction on the number of heads painted with a blue spot.

Base case: Suppose the bishop only painted 1 monk's head with a blue spot. When that monk looks around at all the other monks, he doesn't see any blue spots on their heads. Since the bishop said that at least one monk has a blue spot on his head, the monk realizes that he is the monk. He leaves that night, after the 1st night.

Induction hypothesis: Suppose that if n monks are given a blue spot on their heads, they all leave on the nth night.

Induction step: When n+1 monks with blue spots on their heads look around, each one counts n monks with a blue spot on their head. They all wait for the nth night to come and go and see if the other n monks figure it out and leave (by the induction hypothesis). Since they're all waiting for the others to go (they haven't figured it out, yet) everyone is still there on the (n+1)st day. Each of the n+1 monks realizes that the other n monks saw another monk with a blue spot and since he sees no other blue spots, the monk himself must have a blue spot on his head. Thus all n+1 monks figure it out on the (n+1)st day and they all leave on the (n+1)st night.

In particular, the 10 monks leave on the 10th day.

Now, what happens if the bishop does not mention that at least one of them has a blue spot? The interesting thing is that all 10 monks know that at least one person has a blue spot, since there are 9 monks with a blue spot! And yet, none of them will figure out that they have a blue spot on their head and will never leave. Try thinking about why this is.