MAT 319: MIDTERM 2
SOLUTIONS

1. Determine whether each of the following statements is true or false, and circle your answer. No explanations or justifications are needed.

(a) The Monotone Convergence Theorem says that every monotone sequence converges.

FALSE

(b) If (x_n) converges to 0, there exists a subsequence (x_{n_k}) that converges to 1.

FALSE

(c) The Squeeze Theorem says that if (x_n), (y_n), and (z_n) are three sequences, $x_n \leq y_n \leq z_n$, and $\lim x_n = a$, $\lim z_n = b$, then $a \leq \lim y_n \leq b$.

FALSE

(d) If $\lim x_n = +\infty$, $\lim y_n = -\infty$, then $\lim(x_n + y_n) = 0$.

FALSE

(e) Every bounded sequence has a convergent subsequence.

TRUE

2. Determine whether the following sequences are convergent or divergent, and justify your answer. You may use any theorems from the course, but you have to give a clear reference and explain exactly how you apply the theorem.

(a) $x_n = \dfrac{n^2+1}{1+3n-n^2} = -1.$

SOLUTION: $x_n = \dfrac{n^2+1}{1+3n-n^2} = \dfrac{1+1/n^2}{1/n^2+3/n-1}$. We know that $\lim \frac{1}{n} = 0$, $\lim 1 = 1$, $\lim 3 = 3$. By the Product Rule for limits, $\lim 1/n^2 = 0$ and $\lim 3/n = 0$. By the Sum Rule, $\lim(1+1/n^2) = 1+0 = 1$, $\lim(1/n^2+3/n-1) = 0+0-1 = -1 \neq 0$. By the Quotient Rule, $\lim x_n = 1/(-1) = -1$.

(b) $y_n = \dfrac{\cos(n^2)}{n^2} = 0$.
SOLUTION: Observe that $-1 \leq \cos(n^2) \leq 1$. Then $-1/n^2 \leq \cos(n^2)/n^2 \leq 1/n^2$. Because $\lim 1/n^2 = \lim -1/n^2 = 0$, $\lim y_n = 0$ by the Squeeze theorem.

3. (a) Suppose that the sequence (x_n) does NOT converge to 5. State what this means in terms of "ε, K". (Construct a negation for the definition of the limit.)
SOLUTION: There exists $\varepsilon > 0$ (" a bad epsilon") such that for every $K \in \mathbb{N}$ there exists $n \geq K$ such that $|x_n - 5| \geq \varepsilon$.

(b) Let $y_n = 3 + (-1)^n 2$. Arguing from definitions, show that (y_n) does not converge to 5.
SOLUTION: Set $\varepsilon = 1$ and show that this is a "bad epsilon" from the above statement. Indeed, for every $K \in \mathbb{N}$ there exist an odd number $n \geq K$ (e.g. $n = K$ or $n = K+1$). We have $y_n = 3 + (-1)^n 2 = 3 - 2 = 1$, and so $|y_n - 5| = 4 > \varepsilon = 1$.

4. Suppose that $\lim x_n = +\infty$. Show that (x_n) is bounded below.
SOLUTION: Fix an arbitrary $\alpha \in \mathbb{R}$, say $\alpha = 0$. By definition, $\lim x_n = +\infty$ means that there exists $K \in \mathbb{N}$ such that $x_n > \alpha$ for all $n \geq K$. Therefore, the set $\{x_K, x_{K+1}, x_{K+2}, \dots\}$ is bounded below by α. The set $\{x_1, x_2, \dots, x_{K-1}\}$ is finite, and therefore bounded below by $m = \min\{x_1, x_2, \dots, x_{K-1}\}$. It follows that (x_n) is bounded below by $\min(\alpha, m)$.

5. Let the sequence (x_n) be defined by $x_1 = 6$, $x_{n+1} = \frac{2}{3}x_n + 1$ for $n \in \mathbb{N}$. Prove that (x_n) converges, and find its limit.
SOLUTION: We will prove that (x_n) is decreasing and bounded below. Then it has a limit by monotone convergence theorem.

First, observe that $x_1 > 0$, and that x_n remains positive for all n, so 0 is a lower bound.

Prove that $x_n > x_{n+1}$ by induction . Base: $x_1 = 6 > x_2 = 5$. Induction step: let n be arbitrary, and assume that $x_n > x_{n+1}$. Then $x_{n+1} = 2/3x_n + 1 > 2/3x_{n+1} + 1 = x_{n+2}$. It follows that the sequence decreases.

Now, suppose $\lim x_n = L$. Then $\lim x_{n+1} = L$ (a tail has the same limit); on the other hand, $\lim x_{n+1} = \lim(2/3x_n + 1) = 2/3L + 1$. Solve $L = 2/3L + 1$, get $L = 3$.

6. Let (x_n) be a sequence such that $\lim x_n = 0$. Consider the sequence (y_n), where

$$y_{2n-1} = x_n, \quad y_{2n} = \frac{1}{n} \qquad \text{for all } n \in \mathbb{N}.$$

(The first few terms of this sequence are $x_1, 1, x_2, \frac{1}{2}, x_3, \frac{1}{3}, \ldots$) Using the definition of the limit, prove that the sequence y_n converges and find its limit.

SOLUTION: Obviously, both subsequences (y_{odd}) and (y_{even}) tend to 0. By definition of limit, this means that for every $\epsilon > 0$, there exist $K_1, K_2 \in \mathbb{N}$ such that $|y_{2n-1} - 0| < \epsilon$ for all n such that $2n - 1 \geq K_1$, and also $|y_{2n} - 0| < \epsilon$ for all n such that $2n \geq K_2$. Set $K = \max(K_1, K_2)$, then $|y_n - 0| < \epsilon$ for all $n \geq K$.

7. Show that if (x_n) tends to $-\infty$, then $x_n + 10$ also tends to $-\infty$.

SOLUTION: By definition, we must show that for every α there exists $K \in \mathbb{N}$ such that $x_n + 10 < \alpha$ for all $n \geq K$. Because (x_n) tends to $-\infty$, for $\beta = \alpha - 10$ there exists $K' \in \mathbb{N}$ such that $x_n < \beta = \alpha - 10$ for all $n \geq K'$. Now for $K = K'$ we have $x_n + 10 < \alpha - 10 + 10 = \alpha$ for all $n \geq K$.