MAT 319/320: REVIEW SHEET FOR MIDTERM 1

The exam will cover the material of Chapters 1 and 2. You are expected to know the basic definitions and how to apply them in straightforward examples and arguments. In detail:

- you should know all the definitions in Ch 1 and Ch 2. In the exam we will ask you to state one or two of these.
- we do not expect you to memorize the 9 properties of a field (2.1.1) or the derivation in 2.1 of all the basic properties of the algebraic operations and order on $\mathbb{R}$.
- you should be able to state the important Properties:
  (1.2.1) Well-Ordering Property of $\mathbb{N}$
  (2.3.6) Completeness property of $\mathbb{R}$
  (2.4.4) Archimedean Property
  (2.5.2) Nested Intervals Property
- you should know the statements of the most important theorems and have a good idea of their proofs. We will not ask you to reproduce any of these proofs, but we will expect you to be able to make short arguments using the basic theorems and properties.

The best way to prepare for the exam is to go over all the homeworks. On the next page, there are some additional practice problems.

Here are some sample problems for the exam. Some of the questions are too long for the actual exam, and would be shortened. There are more than 5 questions here, to give you more practice.

1. (a) Show that if $f\colon A \to B$ is injective and $E \subseteq A$ then $f^{-1}(f(E)) = E$.
(b) Give an example to show that equality need not hold if f is not injective.

2. Prove the following statement by induction:

$$3 + 11 + \cdots + (8n - 5) = 4n^2 - n \qquad \text{for all } n \in \mathbb{N}.$$

3. (a) Show that the following statement is false by giving a counterexample:

$$\forall\, x \in \mathbb{R} \exists\, t \in \mathbb{R} : \frac{t^2}{1-x} > 1$$

(b) Prove the following statement by contradiction (indirect proof)

$$\sqrt{2} + \sqrt{3} > \sqrt{5}.$$

Note: here we denote by $\sqrt{x}$ the positive square root of a positive real number x. You may use all the standard results on the order properties of the real numbers without proof.

4. (a) What does it mean for a set to be countable?
(b) Show from the definition that the set $\mathbb{Q}$ of rational numbers is countable.
(c) Show from the definition that if S and T are countable then so is $S \cup T$. First consider the case when S and T are disjoint ($S \cap T = \varnothing$), and then the general case.

5. State the Archimedean property of the real numbers. Use it to prove that if $x > 0$ is any real number then there is a rational number r such that $x > r > 0$.

6. (a) Let S be a subset of $\mathbb{R}$. Give the definitions of a lower bound of S and of $\inf S$.
(b) If possible, give examples of sets S with the following properties. If there is no such example, give a brief explanation of why.
(i) a set $S \subset \mathbb{R}$ with no lower bound.
(ii) a set $S \subset \mathbb{R}$ with a lower bound but no infimum.