MAT 319: MIDTERM 1

SOLUTIONS

1. Determine whether each of the following statements is true or false, and circle your answer. No explanations or justifications are needed.

(a) The Archimedean property says that for every real number x there exists a natural number n such that $x < n$.

$\boxed{\text{TRUE}}$ FALSE

(b) Let A be a countable set and B, an uncountable set. Then $A \cup B$ is uncountable.

$\boxed{\text{TRUE}}$ FALSE

(c) For every two real numbers x, y, $|x| - |y| \leq |x - y|$.

$\boxed{\text{TRUE}}$ FALSE

(d) Every real number has a unique decimal representation.

TRUE $\boxed{\text{FALSE}}$

(e) Every set bounded from below has a supremum.

TRUE $\boxed{\text{FALSE}}$

2. Let $f : X \to Y$ be a map, $A \subset X$, $B \subset Y$.

(a) Is it always true that $f^{-1}(f(A)) = A$?
Prove or give a counterexample.
Solution: The statement is false. Counterexample: let $f(x) = x^2$ and $A = \{1\}$. Then $f(A) = \{1\}$ but $f^{-1}(f(A)) = \{-1, 1\} \neq A$.

(b) Is it always true that $f(f^{-1}(B)) = B$?
Prove or give a counterexample.
Solution: This is also false. Counterexample: let $f(x) = x^2$ and $B = \{-1\}$. Then $f^{-1}(B) = \emptyset$ and $f(f^{-1}(B)) \neq B$.

3. Prove that for every natural number n,

$$1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n + 1)! - 1.$$

Solution: Induction base: for $n = 1$, $1 \cdot 1! = 2! - 1$.

Induction step: assume the statement holds for $n = k$. Then $1 \cdot 1! + 2 \cdot 2! + \cdots + k \cdot k! + (k+1) \cdot (k+1)! = (k+1)! - 1 + (k+1) \cdot (k+1)! = (1+k+1)(k+1)! - 1 = (k+2)(k+1)! - 1 = (k+2)! - 1$, hence the statement holds for $n = k + 1$.

4. For this question, you can use all the algebraic properties of $\mathbb{R}$ without any explanations. You must, however, justify everything that concerns the order properties.

Recall that the set of *positive* numbers $\mathbb{P}$ is defined as a nonempty subset of $\mathbb{R}$ that satisfies the following properties:

(i) If $a, b \in \mathbb{P}$, then $a + b \in \mathbb{P}$.

(ii) If $a, b \in \mathbb{P}$, then $ab \in \mathbb{P}$.

(iii) If $a \in \mathbb{R}$, then exactly one of the following holds:

$$a \in \mathbb{P}, \qquad a = 0, \qquad -a \in \mathbb{P}.$$

We also showed in class that

(iv) $1 \in \mathbb{P}$.

Also, recall that by definition $x > y$ if and only if $x - y \in \mathbb{P}$.

(a) Using properties (i)–(iv), show that if $a \in \mathbb{P}$, then $1/a \in \mathbb{P}$.

Solution: Assume that $1/a \notin \mathbb{P}$. Then by (iii) either $1/a = 0$ or $-1/a \in \mathbb{P}$. In the former case, we have $1 = a\cdot(1/a) = a\cdot 0 = 0$, contradiction. In the latter case, $a \cdot (-1/a) \in \mathbb{P}$ by (ii). Hence $-1 \in \mathbb{P}$. By (iv), we also have $1 \in \mathbb{P}$, which contradicts (iii).

(b) Using properties (i)–(iv) and the definition for $x > y$, show that if $u, v \in \mathbb{P}$ and $u > v$, then $1/v > 1/u$.

Solution: We have to prove that $1/v - 1/u \in \mathbb{P}$. Since

$$\frac{1}{v} - \frac{1}{u} = \frac{u-v}{uv} = (u-v)\frac{1}{uv},$$

by (ii) it suffices to prove that $u - v \in \mathbb{P}$ and $1/uv \in \mathbb{P}$. The first statement follows because $u > v$. To show that $1/uv \in \mathbb{P}$, note first that $u, v \in \mathbb{P}$, hence by (ii), $uv \in \mathbb{P}$. By part (a), $1/uv \in \mathbb{P}$.

5. Prove that the set S of all roots of natural numbers,

$$S = \{x \in \mathbb{R} : x = \sqrt[n]{m} \text{ for } m, n \in \mathbb{N}\},$$

is countable.

Solution: Note first that the set S cannot be represented by the set of all pairs (n, m) since each number in S can be represented by different roots. For example, $\sqrt{2} = \sqrt[4]{4}$, $\sqrt[3]{125} = \sqrt{25}$ etc. However, S can be viewed as a subset of $\mathbb{N} \times \mathbb{N}$.

For each $x \in S$ consider all pairs (n, m) such that $x = \sqrt[n]{m}$. Choose the pair with minimal n. Thus for each $x \in S$, we choose exactly one pair (n, m). This establishes a one-to-one correspondence between S and a subset of $\mathbb{N} \times \mathbb{N}$. Since $\mathbb{N} \times \mathbb{N}$ is countable, so is S.

Another **Solution:** Use the diagonal method to enumerate elements of S. Just as in the solution above, we have to be careful about choosing exactly one pair (m, n) to represent an element of S and avoid pairs (m, n) that represent a root $\sqrt[n]{m}$ that we already counted.

6. Let $f : [0, 1] \to \mathbb{R}$ be a bounded function.
Define the function $g : [0, 1] \to \mathbb{R}$ by $g(x) = -f(x)$.
Show that if $\sup_{0 \le x \le 1} f(x) = a$, then $\inf_{0 \le x \le 1} g(x) = -a$.

Solution: First of all, we have to show that $-a$ is a lower bound of $g(x)$. Let $x \in [0, 1]$. Then $f(x) \le a$. Hence $-f(x) \ge -a$, i.e. $g(x) \ge -a$ making $-a$ a lower bound.

Now, let b be a lower bound of $g(x)$ such that $b > -a$. Then $g(x) \ge b$ for all $x \in [0, 1]$. Hence $f(x) \le -b$. But $-b < a$ which contradicts a being the supremum of $f(x)$.

7. (a) State the Nested Intervals Property.

Solution: Let I_n, $n = 1, 2, 3 \ldots$ be a sequence of nested ($I_{n+1} \subseteq I_n$) closed, bounded intervals. Then $\bigcap_{n=1}^{\infty} I_n$ is non-empty. [The **closed, bounded** condition is very important. Without it, the theorem is not true.]

(b) Give an example of a sequence I_n, $n = 1, 2, 3 \ldots$ of nested open, bounded intervals, such that

$$\bigcap_{n=1}^{\infty} I_n = \emptyset.$$

(Prove that the intersection is empty).

Solution: Let $I_n = (0, 1/n)$. Assume there exists $c \in \bigcap_{n=1}^{\infty} I_n$. Since $c \in I_n$ for all n, c certainly lies in, for instance, I_{17}. Thus $0 < c < 1/17$ and c is positive. Now, by the Archimedean Property, there exist a natural number N such that $1/c < N$. Then $c > 1/N$ and $c \notin I_N$, contradiction.